PHYS 340 THERMAL PHYSICS WQ 2011

Midterm Exam 2

1 Free energy of a two state system (2+2+2+2 points)

Consider a system with two states, one at zero energy and one at energy ε.

a) Find the free energy as a function of the temperature.
b) Determine the internal energy of the system.
c) What is the entropy of the system?
d) Evaluate σ for very large temperatures.

2 Harmonic oscillator (2+2+2 points)

The partition function of a harmonic oscillator is: $Z = \dfrac{1}{1-e^{-\hbar\omega_n/\tau}}$

a) Calculate the average energy of the system.
b) Determine the free energy of the system.
c) Evaluate the entropy of the system.

3 Free energy of a photon gas (2+1+4 points)

a. Show that the partition function of a photon gas is given by

$$Z = \prod_n \frac{1}{1-e^{-\hbar\omega_n/\tau}}$$

, where the product runs over all possible modes of the electromagnetic field characterized by their frequencies ω_n.

b. Write down the free energy of the system as a sum.

c. If one converts the sum to an integral and integrates by parts, one finds

$$F = -\frac{\pi^2 V \tau^4}{45\hbar^3 c^3}.$$

Use this expression to calculate the entropy and the internal energy of the photon gas.

4 Proof (3 points)

Prove the Maxwell relation: $\left(\dfrac{\partial\tau}{\partial V}\right)_\sigma = -\left(\dfrac{\partial p}{\partial\sigma}\right)_V$

using second partial derivatives of the internal energy. (*Hint: Note that* $\tau = (\partial U / \partial \sigma)_V$ *and use the definition of pressure as the rate of change of energy with volume.)*

Phys 340

Midterm Exam 1

1) Free energy of a two-State system

$$Z = \sum_s e^{-\varepsilon_s/\tau} = 1 + e^{-\varepsilon/\tau}$$

a) $$F = -\tau \log Z = \underline{\underline{-\tau \log\left(1 + e^{-\varepsilon/\tau}\right)}}$$

b) $$U = \tau^2 \frac{\partial \log Z}{\partial \tau} = \tau^2 \frac{1}{1 + e^{-\varepsilon/\tau}} e^{-\varepsilon/\tau} \left(\frac{\varepsilon}{\tau^2}\right)$$

$$= \underline{\underline{\frac{\varepsilon}{e^{\varepsilon/\tau} + 1}}}$$

c) $$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = \log\left(1 + e^{-\varepsilon/\tau}\right) + \tau \frac{e^{-\varepsilon/\tau}}{1 + e^{-\varepsilon/\tau}} \left(\frac{\varepsilon}{\tau^2}\right)$$

$$= \underline{\underline{\log\left(1 + e^{-\varepsilon/\tau}\right) + \frac{\varepsilon/\tau}{e^{+\varepsilon/\tau} + 1}}}$$

As $\tau \to \infty$ $$\sigma_{\tau \to \infty} = \log(1+1) + \frac{0}{2} = \underline{\underline{\log 2}}$$

3) Harmonic Oscillator

$$Z = \frac{1}{1-e^{-\hbar\omega/\tau}}$$

a) Average energy: $\langle \varepsilon \rangle = U = \tau^2 \frac{\partial \log Z}{\partial \tau} = \frac{\tau^2}{Z}\frac{\partial Z}{\partial \tau}$

$$= \frac{\cancel{\tau^2}}{(1-e^{-\hbar\omega/\tau})^{-1}}\cancel{(-1)}\frac{1}{(1-e^{-\hbar\omega/\tau})^2}(-\hbar\omega)\frac{\cancel{(-1)}}{\cancel{\tau^2}}e^{-\hbar\omega/}$$

$$= \frac{-\hbar\omega\, e^{-\hbar\omega/\tau}}{1-e^{-\hbar\omega/\tau}}$$

$$= \underline{\underline{\frac{\hbar\omega}{e^{+\hbar\omega/\tau}-1}}}$$

b) Free energy: $F = -\tau \log Z = \underline{\underline{\tau \log\left(1-e^{-\hbar\omega/\tau}\right)}}$

c) Entropy: $\sigma = \frac{1}{\tau}(U-F)$

$$= \underline{\underline{\frac{\hbar\omega/\tau}{e^{\hbar\omega/\tau}-1} - \log\left(1-e^{-\hbar\omega/\tau}\right)}}$$

3) a) Partition function of photon gas: $$Z_\omega = \sum_{s=0}^{\infty} e^{-\varepsilon_s/\tau} = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau}$$

$$= \sum_{s=0}^{\infty} \left(e^{-\hbar\omega/\tau}\right)^s = \frac{1}{1-e^{-\hbar\omega/\tau}}$$

This is the partition function for a photon of frequency ω, so the total partition function is the product of these:

$$Z = \prod_{\omega} Z_\omega \equiv \prod_n \frac{1}{1-e^{-\hbar\omega_n/\tau}}$$

3) b) $$F = -\frac{\pi^2 V \tau^4}{45\hbar^3 c^3}$$

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = +\frac{\pi^2 V}{45\hbar^3 c^3} 4\tau^3 = \underline{\underline{\frac{4\pi^2 V}{45\hbar^3 c^3}\tau^3}}$$

$$F = U - \tau\sigma \Rightarrow U = F + \tau\sigma$$

$$= \frac{\pi^2 V}{45\hbar^3 c^3}\tau^4(-1+4) = \underline{\underline{\frac{\pi^2 V}{15\hbar^3 c^3}\tau^4}}$$

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b) $$F = -\tau \log Z = -\tau \log \prod_n \left(\frac{1}{1-e^{-\hbar\omega_n/\tau}}\right) = +\tau \sum_n \log\left(1-e^{-\hbar\omega_n/\tau}\right)$$