Coffee Hour Problems and Solutions

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Week 1. *Proposed by Matthew McMullen.*

An interesting fact about a 5-12-13 right triangle is that its area and perimeter are numerically equal. Find a triangle (not necessarily right) with integer side-lengths whose area and perimeter are both numerically equal to 42.

Solution. Let a, b, and c be the side-lengths of the triangle we are trying to find; where, without loss of generality, $a \leq b \leq c$. We are given that $a+b+c = 42$ and, using Heron's formula,

$$\sqrt{21(21-a)(21-b)(21-c)} = 42,$$

which is equivalent to

$$(21-a)(21-b)(21-c) = 84 = 2^2 \cdot 3 \cdot 7.$$

Put $u = 21 - a$, $v = 21 - b$, and $w = 21 - c$. Then $0 < w \leq v \leq u$ and $uvw = 84$. The only possible positive-integer triples (w, v, u) are $(3, 4, 7)$, $(2, 3, 14)$, $(2, 6, 7)$, $(1, 2, 42)$, $(1, 3, 28)$, $(1, 4, 21)$, $(1, 6, 14)$, and $(1, 7, 12)$. But we also need to satisfy $u + v + w = 63 - (a + b + c) = 21$, so the only triple that works is $(1, 6, 14)$. Thus, our triangle has sides of length 7, 15, and 20.

Week 2. *Proposed by Matthew McMullen.*

Show that

$$\sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2\sin(x/2)$$

for all x with $0 \leq x \leq \pi/2$, but not for any x with $\pi/2 < x < 2\pi$.

Solution. Notice first that $\sqrt{1 + \sin x} \leq \sqrt{1 - \sin x}$ if and only if $\sin x \leq 0$. Thus, when $\pi \leq x < 2\pi$, the left-hand side of the equation is at most 0 and the right-hand side is positive. Now, suppose $0 \leq x < \pi$. In this case (since the left-hand side is positive), we can square both sides of our equation to get the equivalent equation

$$2 - 2\sqrt{1 - \sin^2 x} = 4\sin^2(x/2),$$

or

$$\frac{1 - |\cos x|}{2} = \sin^2(x/2).$$

When $0 \le x \le \pi/2$, $\cos x \ge 0$ and the above reduces to the familiar identity

$$\frac{1 - \cos x}{2} = \sin^2(x/2).$$

When $\pi/2 < x < \pi$, we would have

$$\frac{1 + \cos x}{2} = \sin^2(x/2) = \frac{1 - \cos x}{2},$$

which is only true when $\cos x = 0$ (which doesn't happen between $\pi/2$ and π).

Week 3. *Proposed by Matthew McMullen.*

Find

$$\int_1^\infty \frac{\ln(x-1)}{x^{3/2}}\, dx.$$

Solution (outline). Using integration by parts with $u = \ln(x-1)$ and $dv = x^{-3/2}\, dx$, we have that

$$\int \frac{\ln(x-1)}{x^{3/2}}\, dx = \frac{-2\ln(x-1)}{\sqrt{x}} + 2\int \frac{1}{\sqrt{x}(x-1)}\, dx.$$

We can use the substitution $w = \sqrt{x}$ followed by partial fraction decomposition to get

$$\int \frac{1}{\sqrt{x}(x-1)}\, dx = \int \left(\frac{1}{w-1} - \frac{1}{w+1}\right) dw.$$

Thus,

$$\int_1^\infty \frac{\ln(x-1)}{x^{3/2}}\, dx = -2\left(\frac{\ln(x-1)}{\sqrt{x}} - \ln\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)\Bigg|_1^\infty.$$

As x goes to ∞, the above expression goes to 0, so our answer is

$$2\lim_{x\to 1^+}\left(\frac{\ln(x-1)}{\sqrt{x}} - \ln\frac{\sqrt{x}-1}{\sqrt{x}+1}\right).$$

We can rewrite the expression in the above limit as

$$\frac{(\sqrt{x}+1)\ln(\sqrt{x}+1)}{\sqrt{x}} - \frac{(\sqrt{x}-1)\ln(\sqrt{x}-1)}{\sqrt{x}}.$$

Notice that, as x goes to 1 from the right, the first term goes to $2\ln 2$ and the second term goes to 0. Therefore, our answer is $4\ln 2$.

Week 4. *Proposed by Matthew McMullen.*

You roll three fair dice. What is the probability that some subset of the numbers rolled sums to 4? (This includes rolls such as 1,2,1 and 4,5,6.)

Solution (idea). We will count the number of combinations, out of the 6^3 total combinations, that "win". $6^3 - 5^3 = 91$ contain at least one 4. Of those remaining, if the first roll is a 5 or 6, then the next two rolls must be either a 1 and a 3 (in either order), or two 2s. This gives us 6 more winning rolls. In a similar way, we can go through all of the possibilities of winning (that haven't already been counted) if a 3, 2, or 1 is rolled first. There are 10 new ways to win if a 3 is rolled first, 12 new ways to win if a 2 is rolled first, and 12 new ways to win if a 1 is rolled first. The probability we want is therefore

$$\frac{91+6+10+12+12}{6^3} = \frac{131}{216}.$$

Week 5+. *Proposed by Matthew McMullen.*

You roll n fair dice. What is the probability that some subset of the numbers rolled sums to k, where $1 \le k \le 6n$?