# Coffee Hour Problems and Solutions

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**Week 1.** *Proposed by Matthew McMullen.*

Find

$$\lim_{n\to\infty}\int_1^e \frac{n(\ln x)^n}{x^2}\,dx.$$

**Solution.** Using the substitution $u = \ln x$ and the Maclaurin series for $y = e^{-u}$ gives us

$$\begin{aligned}
\int_1^e \frac{n(\ln x)^n}{x^2}\,dx &= \int_0^1 nu^n e^{-u}\,du \\
&= \int_0^1 \sum_{k=0}^{\infty} \frac{n(-1)^k u^{k+n}}{k!}\,du \\
&= \sum_{k=0}^{\infty} \frac{n}{k+n+1}\frac{(-1)^k}{k!}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\int_1^e \frac{n(\ln x)^n}{x^2}\,dx &= \lim_{n\to\infty}\sum_{k=0}^{\infty} \frac{n}{k+n+1}\frac{(-1)^k}{k!} \\
&=^{1} \sum_{k=0}^{\infty} \lim_{n\to\infty}\frac{n}{k+n+1}\frac{(-1)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \\
&= e^{-1}.
\end{aligned}$$

[1] Of course, some justification is needed for this step!

**Week 2.** *Proposed by Matthew McMullen.*

In the Powerball lottery, five numbers from 1 to 69 are chosen (without replacement) and then one "Powerball number" from 1 to 26 is chosen (which may or may not match one of the first five numbers). To win the grand prize, you have to match the first five numbers (in any order) and the Powerball number. If the probability that at least one ticket wins the grand prize is 0.75, how many tickets were sold? (And what assumptions are you using to answer this question?)

**Solution.** There are a total of

$$\frac{69 \cdot 68 \cdot 67 \cdot 66 \cdot 65}{5!} \cdot 26 = 292,201,338$$

different combinations of numbers that can be chosen; so $p = \frac{1}{292{,}201{,}338}$ is the probability of winning the grand prize. Let $n$ be the number of tickets that were sold. We will assume that the number of grand prize-winning tickets (out of $n$) follows a binomial distribution.[2] We are given a 0.25 probability of no winning tickets, which means that

$$(1-p)^n = 0.25,$$

or

$$n = \frac{\ln(0.25)}{\ln(1-p)} \approx \boxed{405{,}076{,}808}.$$

**Week 3.** *Proposed by Matthew McMullen.*

Before the recent \$1.6 billion Powerball drawing, it was reported that approximately 86% of all possible combinations were chosen. Assuming that this means that there is an 86% chance that at least one ticket will win the grand prize, find the probability that there are exactly three grand prize-winning tickets (which is what actually happened).

**Solution.** We first use the same technique (and assumption) as in the previous solution to estimate the number of tickets sold, $n$. For $p = \frac{1}{292{,}201{,}338}$, we have to solve

$$(1-p)^n = 0.14;$$

so

$$n = \frac{\ln(0.14)}{\ln(1-p)} \approx \boxed{574{,}500{,}440}.$$

Now, since $n$ is large and $np \approx 1.966$ is relatively small, the probability we want is well-approximated by a Poisson distribution with mean 1.966.[3] The

[2] Is this a realistic assumption?

[3] Good luck trying to find exact probabilities using a binomial distribution!

probability that there are exactly three grand prize-winning tickets is therefore approximately

$$\frac{1.966^3 e^{-1.966}}{3!} \approx 0.177.$$

(It's interesting to note that this is slightly more than the probability of having no grand prize-winning tickets. On the other hand, the probability of having *at least* three grand prize-winning tickets can be shown to be approximately 0.314.)

**Week 4.** *Proposed by Matthew McMullen.*

Define $a_n$ to be the number formed by concatenating 100,000 $n$ times (for example, $a_3 = 100,000,100,000,100,000$). Find all $k$ such that 2016 divides $a_k$.

**Solution.** Write

$$a_k = 10^5 \sum_{n=0}^{k-1} 10^{6n}.$$

Clearly, $2^5$ divides each $a_k$. Also, looking modulo 9, we have

$$a_k \equiv \sum_{n=0}^{k-1} 1 = k.$$

Thus, $a_k$ is divisible by 9 if and only if $k$ is divisible by 9. Looking modulo 7, we have

$$10^5 \equiv 3^5 = 3 \cdot 9^2 \equiv 3 \cdot 2^2 = 12 \equiv 5,$$

and

$$10^6 = 10 \cdot 10^5 \equiv 3 \cdot 5 = 15 \equiv 1;$$

so,

$$a_k \equiv 5 \sum_{n=0}^{k-1} 1 = 5k.$$

Since 5 and 7 are relatively prime, $a_k$ is divisible by 7 if and only if $k$ is divisible by 7.

Since 7 and 9 are relatively prime, we can put all of this together to see that $a_k$ is divisible by $2016 = 2^5 \cdot 7 \cdot 9$ if and only if $k$ is a positive integer multiple of 63. (Fun fact: $a_{63}/2016$ has 374 digits.)

**Week 5.** *Proposed by Matthew McMullen.*

Find the number of positive-integer ordered pairs $(a, b)$ such that both $a$ and $b$ are less than 100, $ab \leq 500$, and $ab$ is divisible by 100.

**Solution (outline).** The idea is to count the number of divisors of $100k$, for $k = 1, 2, 3, 4, 5$, that are between $k$ and 100, exclusive. When $k = 1$ we get 7 such divisors, when $k = 2$ we get 8 such divisors, when $k = 3$ we get 12 such divisors, when $k = 4$ we get 9 such divisors, and when $k = 5$ we get 4 such divisors. So the answer is $7 + 8 + 12 + 9 + 4 = \boxed{40}$.

**Week 6.** *Proposed by Matthew McMullen.*

For $n \geq 1$, define $a_n$ to be the number of ordered pairs $(a, b)$ of positive integers less than $n$ with the property that $n$ divides $ab$. Show that $a_n$ is odd if and only if $n$ is a multiple of 4.

**Solution.** For $a \neq b$, if $(a, b)$ is a valid pair, then $(b, a)$ is another valid pair. Therefore, the only way we can have an odd number of total pairs is if there are an odd number of valid pairs of the form $(a, a)$. In other words, $a_n$ is odd if and only if there is an odd number of perfect squares in the set $S_n = \{n, 2n, 3n, \ldots, (n-1)n\}$.

Suppose $n = dc^2$, where $d$ is square free (or 1). Then $ln$ is a perfect square only when $l = dm^2$, for some $m$. In order for $ln$ to be in $S_n$, however, $dm^2 = l < n = dc^2$; i.e., $m = 1, 2, \ldots, c-1$. Thus, the number of perfect squares in $S_n$ is $c - 1$.

We then have that $a_n$ is odd if and only if $n = dc^2$, where $d$ is square free (or 1) and $c$ is even. But $c$ is even if and only if $n$ is a multiple of 4.

**Week 7.** *Proposed by Matthew McMullen.*

Let $P$ be a polynomial of degree $k$ that has $n$ independent variables. Find the maximum number of terms $P$ can have.

**Solution (outline).** The answer is $\binom{n+k}{k}$, which we will prove using induction on $k$. This is clearly true for $k = 0$. One can show (not sure of the best way to do it, but it's definitely true!) that the maximum number of terms of degree $k$ is $\binom{n+k-1}{k}$, for all $k \geq 0$.

Now suppose that the maximum number of terms a degree $k$ polynomial in $n$ independent variables can have is $\binom{n+k}{k}$. Then the maximum number of terms a degree $k+1$ polynomial in $n$ independent variables can have is

$$\binom{n+k}{k} + \binom{n+k}{k+1} = \binom{n+k+1}{k+1}.$$

**Weeks 8 and 9.** *Proposed by Matthew McMullen.*

For integers $k$ and $n$ with $1 \leq k \leq n$, define $d^*(n,k)$ to be the number of divisors of $kn$ in the interval $[k,n]$. Show that

$$\sum_{k=1}^{n} \gcd(n,k) = \sum_{k=1}^{n} d^*(n,k).$$

**"Solution".** The left-hand side of the above equality is well-studied and is known to be multiplicative with

$$\sum_{k=1}^{p^r} \gcd(p^r,k) = p^{r-1}((p-1)r+p)$$

for prime $p$. (Thus, we have a closed-form (but ugly!) representation for the gcd-sum.) With some routine (but ugly!) work, we can show that

$$\sum_{k=1}^{p^r} d^*(p^r,k) = p^{r-1}((p-1)r+p)$$

for prime $p$. It remains only to show that the right-hand side is multiplicative, and if you have any suggestions for how to do this, let me know!

**Week 10.** *From the 2016 AIME I.*

For integers $a$ and $b$ consider the complex number

$$\frac{\sqrt{ab+2016}}{ab+100} - \left(\frac{\sqrt{|a+b|}}{ab+100}\right) i.$$

Find the number of ordered pairs of integers $(a,b)$ such that this complex number is a real number.

**Solution.** Let $z$ denote the complex number in question. Suppose that $ab + 2016 \geq 0$. Then $z$ exists and is real if and only if $a = -b$ and $ab \neq -100$. In other words, $z$ exists and is real if and only if $a \neq \pm 10$ and $|a| \leq \lfloor \sqrt{2016} \rfloor = 44$. In this case there are 87 possible values of $a$ (and therefore 87 pairs $(a, b)$): $0, \pm 1, \pm 2, \ldots, \pm 9, \pm 11, \pm 12, \ldots, \pm 44$.

Now suppose that $ab + 2016 < 0$. Then

$$z = \left( \frac{\sqrt{-(ab+2016)} - \sqrt{|a+b|}}{ab+100} \right) i$$

is real only if $\sqrt{-(ab+2016)} = \sqrt{|a+b|}$. So we need either $a+b = ab+2016$ or $a+b = -ab-2016$. The first equation is equivalent to $(a-1)(b-1) = -2015$. By looking at the factors of 2015 and remembering that $ab < -2016$, we can see that there are eight pairs $(a, b)$ that satisfy this equation:

$$(2, -2014), (6, -402), (14, -154), (32, -64), (-64, 32), (-154, 14), (-402, 6), (-2014, 2).$$

Similarly, there are eight pairs that satisfy the equation $a + b = -ab - 2016$.

Putting these two cases together gives us $87 + 8 + 8 = \boxed{103}$ possible pairs.

**Week 11.** *From the 2016 AIME II.*

Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of $a$, $b$, and $c$ is equal to the product of 11, 21, 31, 41, 51, and 61.

**Solution.** See `http://artofproblemsolving.com/wiki/index.php?title=2016_AIME_II_Problems/Problem_8` for a solution.

**Week 13.** *Proposed by Matthew McMullen.*

Let $r > 0$ and let $E$ be the top half of an ellipse centered at the origin and passing through the points $(-r, 0)$ and $(r, 0)$. Define $f(x) = 0$ for $|x| \geq r$ and $f(x) = E$ for $|x| \leq r$. If $f(x)$ is the probability density function for the random variable $X$, find $\mathrm{Var}(X)$.

**Solution.** Suppose that the ellipse goes through the point $(0, s)$. Since $f$ is a probability density function, the area between $E$ and the $x$-axis must be 1. Therefore, $\frac{\pi r s}{2} = 1$, or $s = \frac{2}{\pi r}$. From this we see that, when $|x| \leq r$,

$$f(x) = \frac{2\sqrt{r^2 - x^2}}{\pi r^2}.$$

Now, since $f$ is an even function, $E(X) = 0$; and hence,

$$\begin{aligned}
\mathrm{Var}(X) &= \int_{-r}^{r} x^2 f(x)\,dx \\
&= 2\int_0^r x^2 f(x)\,dx \\
&= \frac{4}{\pi r^2}\int_0^r x^2\sqrt{r^2 - x^2}\,dx \\
&\overset{(x=r\sin\theta)}{=} \frac{r^2}{\pi}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta \\
&\overset{(u=2\theta)}{=} \frac{r^2}{2\pi}\int_0^{\pi} \sin^2 u\,du \\
&= \frac{r^2}{4}.
\end{aligned}$$

**Week** $15^{+}$**.** *Based on A1 on the 2015 Putnam Exam.*

Let $A$ and $B$ be points on the same branch of the hyperbola $xy = 1$. Suppose that $P$ is a point lying between $A$ and $B$ on the hyperbola such that the area of the triangle $APB$ is as large as possible. Show that the region bounded by the hyperbola and the chord $AP$ has the same area as the region bounded by the hyperbola and the chord $PB$.

Conversely, let $f(x)$ be a strictly convex function on some interval $I$ where, for any two points $A$ and $B$ on $f$, the point $P$ between $A$ and $B$ on $f$ that maximizes the area of triangle $APB$ also satisfies the condition that the region bounded by $f$ and the chord $AP$ has the same area as the region bounded by $f$ and the chord $PB$. Is $f$ necessarily the branch of a hyperbola?

**Solution.** See `http://kskedlaya.org/putnam-archive/2015s.pdf` for a solution to the first part. I'm still working on the second part. Let me know if you have any insight!