Coffee Hour Problems and Solutions

Edited by Matthew McMullen

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Week 1. *Proposed by Matthew McMullen.*

We all (hopefully!) know how to rationalize the denominator of an expression like $\frac{1}{\sqrt{2}+\sqrt{3}}$. This week's problem is to rationalize the denominator of $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. As an extra challenge, can you rationalize the denominator of $\frac{1}{\sqrt{2}+\sqrt[3]{3}}$?

Solution. For the first expression, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}} &= \frac{1}{(\sqrt{2}+\sqrt{3})+\sqrt{5}} \cdot \frac{(\sqrt{2}+\sqrt{3})-\sqrt{5}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{(\sqrt{2}+\sqrt{3})^2-5} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\
&= \frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}.
\end{aligned}$$

For the second expression we will use the factorization

$$u^6 - v^6 = (u+v)(u^5 - u^4 v + u^3 v^2 - u^2 v^3 + u v^4 - v^5)$$

and the facts that $\sqrt{2} = 2^{1/2} = 2^{3/6} = 8^{1/6} = \sqrt[6]{8}$ and $\sqrt[3]{3} = 3^{1/3} = 3^{2/6} = 9^{1/6} = \sqrt[6]{9}$. Put $u = \sqrt[6]{8}$, $v = \sqrt[6]{9}$, and $C = u^5 - u^4 v + u^3 v^2 - u^2 v^3 + u v^4 - v^5$. From the above factorization, we have that $(u+v)C = u^6 - v^6 = -1$. Then

$$\begin{aligned}
\frac{1}{\sqrt{2}+\sqrt[3]{3}} &= \frac{1}{u+v} \cdot \frac{C}{C} \\
&= -C \\
&= 3\sqrt[3]{9} - 3\sqrt{2}\sqrt[3]{3} + 6 - 2\sqrt{2}\sqrt[3]{9} + 4\sqrt[3]{3} - 4\sqrt{2}.
\end{aligned}$$

Week 2. *Proposed by Matthew McMullen*

An interesting fact about the numbers 1,2, and 3 is that both their product and sum are equal to 6. Can you find three other rational numbers that satisfy this property? How many different sets of three rational numbers can you find that satisfy this property?

Solution. Suppose $\frac{a}{d}$, $\frac{b}{d}$, and $\frac{c}{d}$ are three rational numbers such that $\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=6$ and $\frac{a}{d}\cdot\frac{b}{d}\cdot\frac{c}{d}=6$. Then $a+b+c=6d$ and $abc=6d^3$. Hoping for the best, we put $d=2$ and look for integers a, b, and c with $a+b+c=12$ and $abc=48$. By inspection, $a=-1$, $b=-3$, and $c=16$ works. Thus, $\{-\frac{1}{2},-\frac{3}{2},8\}$ is one such set.

To generate more sets, we can use techniques from the study of elliptic curves. Suppose $x+y+z=6=xyz$. Then we are looking for rational points on the curve C defined by $xy(6-x-y)=6$. Given a point (x_0,y_0) on C, find the tangent line at that point and look to see where else it intersects C. If (x_0,y_0) is a rational point on C, then this intersection point is another rational point on C.

Leaving off the details, it can be shown that the line tangent to C at $(2,1)$ also intersects C at $(8,-1/2)$. Moreover, the line tangent to C at $(8,-1/2)$ also intersects C at $(-32/323,-361/68)$. So another solution to our problem is the set $\{-\frac{32}{323},-\frac{361}{68},\frac{867}{76}\}$ Continuing in this way, we can find other (infinitely many?!) solutions. Just for fun, the next solution given by this technique is the set

$$\left\{-\frac{14927155328}{32322537971},\frac{79790995729}{9885577384},-\frac{39280614987}{24403407416}\right\}.$$

Week 3. *Proposed by Matthew McMullen*

Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right).$$

Solution (kinda). It can be shown that

$$\sqrt[n]{n!}=\frac{n}{e}+\frac{\ln(2\pi n)}{2e}+\epsilon_n,$$

where $\epsilon_n\to 0$ as $n\to\infty$. Therefore,

$$\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}=\frac{1}{e}+\frac{\ln\left(1+\frac{1}{n}\right)}{2e}+\delta_n,$$

where $\delta_n\to 0$ as $n\to\infty$. So the answer to our problem is $\frac{1}{e}$.

Week 4. *Proposed by Matthew McMullen*

Let (a_n) and (b_n) be sequences of real numbers with $b_n \neq 0$ for all n. Is it true that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ if and only if $\lim_{n\to\infty}(a_n - b_n) = 0$? Is either direction of the implication true? If not, can you find counterexamples for both directions? Can you "fix" the implication by putting more restrictions on (b_n)?

Solution. Neither direction of the implication is true. If $a_n = n+1$ and $b_n = n$, then $\frac{a_n}{b_n} \to 1$, but $a_n - b_n \to 1$. If $a_n = \frac{2}{n}$ and $b_n = \frac{1}{n}$, then $a_n - b_n \to 0$, but $\frac{a_n}{b_n} \to 2$.

Rewriting

$$a_n - b_n = b_n \left(\frac{a_n}{b_n} - 1 \right)$$

and

$$\frac{a_n}{b_n} = \frac{1}{b_n}(a_n - b_n) + 1$$

shows that both directions of the implication will be true as long as both (b_n) and $(1/b_n)$ are bounded sequences.

Week 5. *Proposed by Matthew McMullen*

Let $f(x) = \frac{1-\sqrt{1-4x}}{2x}$. Find $\lim_{x\to 0} f''(x)$. As an added challenge, can you find $\lim_{x\to 0} f^{(n)}(x)$?

Solution. A (messy) way to do this is to rationalize the numerator of f to get $f(x) = \frac{2}{1+\sqrt{1-4x}}$ (for $x \neq 0$), then take the second derivative and plug in 0. In this way, you'll find $\lim_{x\to 0} f''(x) = 4$.

A more elegant way to do this is to find the Taylor Polynomial of $f(x)$ about $x = 0$ and use its coefficients to read off $\lim_{x\to 0} f^{(n)}(x)$ for all n. By repeatedly taking derivatives and plugging in 0, one can show that

$$\sqrt{1-u} = 1 - \sum_{n=0}^{\infty} \frac{2(2n)!u^{n+1}}{4^{n+1}n!(n+1)!}.$$

From this, it is easy to see that

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} x^n.$$[1]

Then $\lim_{x\to 0} f^{(n)}(x) = \frac{(2n)!}{(n+1)!}$.

[1] This is the generating function for the Catalan numbers!

Week 6. *Problem A1 from the 2014 Putnam Competition*

Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Solution. Using the fact that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

it can be shown that

$$(1 - x + x^2)e^x = 1 + \sum_{n=2}^{\infty} \frac{n-1}{n(n-2)!} x^n.$$

For $n - 1$ prime, we're done; so suppose $n - 1 = pk$ for some prime p and some integer $k > 1$. If $k \neq p$, then both k and p will appear in $(n-2)!$, and the numerator of the nth coefficient will be 1. If $k = p$, then at least one p will appear in $(n-2)!$, and the numerator of the nth coefficient will be either 1 or p.

Week 7. *Proposed by Dave Stucki and Matthew McMullen*

Find all real numbers x such that

$$(x^2 - 9x + 17)^{x^2 - 8x + 7} = 1.$$

Solution. The only way this equation can hold is if the base is 1, the base is nonzero and the exponent is 0, or the base is -1 and the exponent is an even integer.

The base is 1 if and only if $x^2 - 9x + 16 = 0$, or $x = \frac{9 \pm \sqrt{17}}{2}$. The exponent is 0 if and only if $0 = x^2 - 8x + 7 = (x-1)(x-7)$, or $x = 1$ or $x = 7$ (and in both of these cases, the base is nonzero). The base is -1 if and only if $0 = x^2 - 9x + 18 = (x-3)(x-6)$. If $x = 3$, we have $(-1)^{-8} = 1$; and if $x = 6$, we have $(-1)^{-5} = -1$.

So the solution set is $\left\{1, 3, 7, \frac{9+\sqrt{17}}{2}, \frac{9-\sqrt{17}}{2}\right\}$.

Week 8. *From Stewart's* Calculus.

A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

Solution. Let a be the depth that the cone is submerged. We are given that $\frac{da}{dt} = 1$. The volume of water that is displaced is given by the formula $V = \frac{1}{3}\pi b^2 a$, where b satisfies the relation $\frac{h}{r} = \frac{a}{b}$. Thus $V = \frac{1}{3}\pi \frac{r^2 a^3}{h^2}$, and we have $\frac{dV}{dt} = \frac{\pi r^2 a^2}{h^2}\frac{da}{dt}$. When the cone is completely submerged, $a = h$, and we have $\frac{dV}{dt} = \pi r^2$.

Let l denote the water level in the cylinder. Then the volume of water in the cylinder is given by $V = \pi R^2 l$, and we have $\frac{dV}{dt} = \pi R^2 \frac{dl}{dt}$. Plugging in the value we found for $\frac{dV}{dt}$, and solving for $\frac{dl}{dt}$, gives us $\frac{dl}{dt} = \frac{r^2}{R^2}$. In other words, the water level is rising at a rate of $\frac{r^2}{R^2}$ cm/s at the instant the cone is completely submerged.

Week 9. *Proposed by Matthew McMullen.*

You are teaching factoring of trinomials and one of your students teaches you the following trick for factoring $6x^2 + 19x + 10$.

First, write $(6x + \_\_\_)(6x + \_\_\_)$, since the leading coefficient is 6. Then, find two numbers that multiply to be 60 (6×10) and add to be 19 (the coefficient of the x term). Put these numbers (4 and 15) in the blanks to get $(6x+4)(6x+15)$. Then throw away any common factors to get $(3x + 2)(2x + 5)$.

Does this trick always work? If not, when will it work?

Solution. Coming soon.