Coffee Hour Problems

Edited by Matthew McMullen

Spring 2016

Week 1. *Proposed by Matthew McMullen.*

Find

$$\lim_{n\to\infty}\int_1^e \frac{n(\ln x)^n}{x^2}\,dx.$$

Week 2. *Proposed by Matthew McMullen.*

In the Powerball lottery, five numbers from 1 to 69 are chosen (without replacement) and then one "Powerball number" from 1 to 26 is chosen (which may or may not match one of the first five numbers). To win the grand prize, you have to match the first five numbers (in any order) and the Powerball number. If the probability that at least one ticket wins the grand prize is 0.75, how many tickets were sold? (And what assumptions are you using to answer this question?)

Week 3. *Proposed by Matthew McMullen.*

Before the recent $1.6 billion Powerball drawing, it was reported that approximately 86% of all possible combinations were chosen. Assuming that this means that there is an 86% chance that at least one ticket will win the grand prize, find the probability that there are exactly three grand prize-winning tickets (which is what actually happened).

Week 4. *Proposed by Matthew McMullen.*

Define a_n to be the number formed by concatenating 100,000 n times (for example, $a_3 = 100,000,100,000,100,000$). Find all k such that 2016 divides a_k.

Week 5. *Proposed by Matthew McMullen.*

Find the number of positive-integer ordered pairs (a, b) such that both a and b are less than 100, $ab \leq 500$, and ab is divisible by 100.

Week 6. *Proposed by Matthew McMullen.*

For $n \geq 1$, define a_n to be the number of ordered pairs (a, b) of positive integers less than n with the property that n divides ab. Show that a_n is odd if and only if n is a multiple of 4.

Week 7. *Proposed by Matthew McMullen.*

Let P be a polynomial of degree k that has n independent variables. Find the maximum number of terms P can have.

Weeks 8 and 9. *Proposed by Matthew McMullen.*

For integers k and n with $1 \leq k \leq n$, define $d^*(n, k)$ to be the number of divisors of kn in the interval $[k, n]$. Show that

$$\sum_{k=1}^{n} \gcd(n, k) = \sum_{k=1}^{n} d^*(n, k).$$

Week 10. *From the 2016 AIME I.*

For integers a and b consider the complex number

$$\frac{\sqrt{ab+2016}}{ab+100} - \left(\frac{\sqrt{|a+b|}}{ab+100}\right) i.$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

Week 11. *From the 2016 AIME II.*

Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of a, b, and c is equal to the product of 11, 21, 31, 41, 51, and 61.

Week 13. *Proposed by Matthew McMullen.*

Let $r > 0$ and let E be the top half of an ellipse centered at the origin and passing through the points $(-r, 0)$ and $(r, 0)$. Define $f(x) = 0$ for $|x| \geq r$ and $f(x) = E$ for $|x| \leq r$. If $f(x)$ is the probability density function for the random variable X, find $\mathrm{Var}(X)$.

Week 15^+**.** *Based on A1 on the 2015 Putnam Exam.*

Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on the hyperbola such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Conversely, let $f(x)$ be a strictly convex function on some interval I where, for any two points A and B on f, the point P between A and B on f that maximizes the area of triangle APB also satisfies the condition that the region bounded by f and the chord AP has the same area as the region bounded by f and the chord PB. Is f necessarily the branch of a hyperbola?