Coffee Hour Problems

Edited by Matthew McMullen

Fall 2015

Week 1. *Proposed by Matthew McMullen.*

We all (hopefully!) know how to rationalize the denominator of an expression like $\frac{1}{\sqrt{2}+\sqrt{3}}$. This week's problem is to rationalize the denominator of $\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. As an extra challenge, can you rationalize the denominator of $\frac{1}{\sqrt{2}+\sqrt[3]{3}}$?

Week 2. *Proposed by Matthew McMullen*

An interesting fact about the numbers 1,2, and 3 is that both their product and sum are equal to 6. Can you find three other rational numbers that satisfy this property? How many different sets of three rational numbers can you find that satisfy this property?

Week 3. *Proposed by Matthew McMullen*

Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right).$$

Week 4. *Proposed by Matthew McMullen*

Let (a_n) and (b_n) be sequences of real numbers with $b_n \neq 0$ for all n. Is it true that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ if and only if $\lim_{n\to\infty}(a_n - b_n) = 0$? Is either direction of the implication true? If not, can you find counterexamples for both directions? Can you "fix" the implication by putting more restrictions on (b_n)?

Week 5. *Proposed by Matthew McMullen*

Let $f(x) = \frac{1-\sqrt{1-4x}}{2x}$. Find $\lim_{x\to 0} f''(x)$. As an added challenge, can you find $\lim_{x\to 0} f^{(n)}(x)$?

Week 6. *Problem A1 from the 2014 Putnam Competition*

Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Week 7. *Proposed by Dave Stucki and Matthew McMullen*

Find all real numbers x such that

$$(x^2 - 9x + 17)^{x^2-8x+7} = 1.$$

Week 8. *From Stewart's* Calculus.

A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

Week 9. *Proposed by Matthew McMullen.*

You are teaching factoring of trinomials and one of your students shows you the following trick for factoring $6x^2 + 19x + 10$.

First, write $(6x + \_\_\_)(6x + \_\_\_)$, since the leading coefficient is 6. Then, find two numbers that multiply to be 60 (6×10) and add to be 19 (the coefficient of the x term). Put these numbers (4 and 15) in the blanks to get $(6x+4)(6x+15)$. Then throw away any common factors to get $(3x + 2)(2x + 5)$.

Does this trick always work? If not, when will it work?