Coffee Hour Problems and Solutions

Edited by Matthew McMullen

Spring 2015

Week 1. *Proposed by Matthew McMullen.*

Show that
$$\arccos\left(\frac{2x}{x^2+1}\right) = \arctan x - \arctan\frac{1}{x}$$
if and only if $x \geq 1$.

Solution. First, since $-1 \leq \frac{2x}{x^2+1} \leq 1$ for all x, the left-hand side always makes sense. If $x = 0$, however, the right-hand side is undefined, so the equation cannot be true then. When $x = 1$, both sides equal 0; but when $x = -1$, the left-hand side is π, while the right-hand side is 0. It remains to check the cases $|x| > 1$ and $0 < |x| < 1$.

For $x \neq 0$ the derivative of the right-hand side can be shown to be $\frac{2}{x^2+1}$. For $x \neq \pm 1$, the derivative of the left-hand side reduces to
$$\frac{2(x^2-1)}{(x^2+1)|x^2-1|}.$$
Thus, when $|x| > 1$ the derivatives of the left- and right-hand sides are equal; but, when $0 < |x| < 1$, these derivatives are not equal. Moreover, when $x = \sqrt{3}$, both sides of the equation equal $\pi/6$; but when $x = -\sqrt{3}$, the left-hand side is $5\pi/6$ and the right-hand side is $-\pi/6$.

Week 2. *Proposed by Matthew McMullen.*

Suppose a_0, a_1, a_2, a_3, and a_4 are all non-negative and satisfy
$$\sum_{n=0}^{4} a_n = 1, \; \sum_{n=0}^{4} n a_n = 1, \; \sum_{n=0}^{4} n^2 a_n = 2, \; \text{and} \; \sum_{n=0}^{4} n^3 a_n = 5.$$
What is the largest possible value of a_0?

Solution. The given system of four equations can be solved by using the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 4 & 9 & 16 & 2 \\ 0 & 1 & 8 & 27 & 64 & 5 \end{array}\right).$$

After row reduction, we obtain the matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 1/3 \\ 0 & 1 & 0 & 0 & 4 & 1/2 \\ 0 & 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1/6 \end{array}\right).$$

Therefore, $a_0 = 1/3 + a_4$, $a_1 = 1/2 - 4a_4$, $a_2 = 6a_4$, and $a_3 = 1/6 - 4a_4$. Since each a_i is non-negative, $0 \leq a_4 \leq 1/24$. Thus, $a_0 \leq 1/3 + 1/24 = \boxed{3/8}$.

Week 3. *Proposed by Matthew McMullen.*

Evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\cot^3 x}\, dx.$$

(Hint: Try the substitution $u = \frac{\pi}{2} - x$.)

Solution. Let I be the integral in question. Multiplying the numerator and denominator of the integrand by $\sin^3 x$ yields

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx.$$

Using the suggested substitution and the cofunction identities for sine and cosine yields

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 u}{\sin^3 u + \cos^3 u}\, du.$$

Therefore,

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\, dx = \frac{\pi}{6},$$

which means that $I = \boxed{\pi/12}$. (Note that this argument works for any power of cotangent in the original integrand.)

Week 4. *Proposed by Matthew McMullen.*

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 2016 \\ 4y + 5z &= 2014 \end{aligned}$$

given that x, y, and z are integers; $z > 0$; and xyz is as small as possible.

Solution. Reducing the last equation modulo 4 tells us that z is two more than a multiple of 4, say $z = 2 + 4n$ for some nonnegative integer n. Solving for x and y in terms of n gives us $x = 1008 - 2n$ and $y = 501 - 5n$. Therefore, we need to minimize

$$f(n) = (1008 - 2n)(501 - 5n)(2 + 4n)$$

for nonnegative integers n. It can be shown that the critical points of f occur at

$$n = \frac{12,074 \pm \sqrt{85,543,036}}{60} \approx 47.1 \text{ and } 355.4.$$

Since f is a cubic polynomial with positive leading coefficient, its minimum value occurs at either $n = 355$ or $n = 356$. When $n = 355$ we have $f(n) = -539,865,144$, and when $n = 356$ we have $f(n) = -539,860,784$. Thus $n = 355$, which means the solution is $\boxed{x = 298,\ y = -1274, \text{ and } z = 1422}$.

Week 5. *Proposed by Matthew McMullen.*

The game of craps involves a shooter repeatedly rolling two fair dice. The simplest bet in this game is the "pass" bet. If the shooter's first roll is a 7 or 11, "pass" wins. If the shooter's first roll is a 2, 3, or 12, "pass" loses. Otherwise, whatever number is rolled first must be rolled again before a 7 is rolled in order for "pass" to win. Wikipedia says that the chance of winning a "pass" bet is $\frac{244}{495}$. Prove that this is correct.

Solution. Let's first find the probability of rolling a 4 before a 7, which is the same as the probability of rolling a 10 before a 7. You can roll a 4 right away, not roll a 4 or 7 and then roll a 4, not roll a 4 or 7 twice and then roll a 4, etc. Thus,

$$P(4 \text{ before } 7) = P(10 \text{ before } 7) = \frac{3}{36}\left(1 + \frac{27}{36} + \left(\frac{27}{36}\right)^2 + \cdots\right) = \frac{3}{36}\cdot\frac{36}{9} = \frac{1}{3}.$$

Similarly,

$$P(5 \text{ before } 7) = P(9 \text{ before } 7) = \frac{4}{36}\left(1 + \frac{26}{36} + \left(\frac{26}{36}\right)^2 + \cdots\right) = \frac{4}{36}\cdot\frac{36}{10} = \frac{2}{5},$$

and

$$P(6 \text{ before } 7) = P(8 \text{ before } 7) = \frac{5}{36}\left(1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \cdots\right) = \frac{5}{36} \cdot \frac{36}{11} = \frac{5}{11}.$$

Now, there are seven different ways of winning. The shooter can roll a 7 or 11 on the first roll, roll a 4 then a 4 before a 7, roll a 10 then a 10 before a 7, roll a 5 then a 5 before a 7, roll a 9 then a 9 before a 7, roll a 6 then a 6 before a 7, or roll an 8 then an 8 before a 7. Thus, the chance of winning a "pass" bet is

$$\frac{8}{36} + 2 \cdot \frac{3}{36} \cdot \frac{1}{3} + 2 \cdot \frac{4}{36} \cdot \frac{2}{5} + 2 \cdot \frac{5}{36} \cdot \frac{5}{11} = \frac{244}{495},$$

as desired.

Week 6. *Proposed by Matthew McMullen.*

You have enough money to place one bet on "pass" in craps. Recall from last week that the probability of winning this bet is $\frac{244}{495}$. The casino pays 1:1 for this bet. You decide to keep betting on "pass" as long as you have enough money to do so. Find the probability that you will be able to place more than nine total bets.

Solution. Let $p = \frac{244}{495}$, and put $q = 1 - p$. Let $P(x)$ denote the probability that you place exactly x bets. Clearly, $P(1) = q$. The only way to place exactly three bets is to win the first bet and lose the next two; so $P(3) = pq^2$. There are two ways to place exactly five bets: win the first two and lose the next three; or win the first, lose the second, win the third, and lose the next two. Thus, $P(5) = 2p^2q^3$. Similarly, there are five ways to place exactly seven bets and fourteen ways to place exactly nine bets. Thus, $P(7) = 5p^3q^4$ and $P(9) = 14p^4q^5$.

The probability of placing more than nine total bets is

$$\begin{aligned} 1 - \sum_{n=1}^{5} P(2n-1) &= 1 - q\left(1 + pq + 2(pq)^2 + 5(pq)^3 + 14(pq)^4\right) \\ &\approx \boxed{0.2355}. \end{aligned}$$

Week 7. *Proposed by Matthew McMullen.*

Suppose that a casino has a "not black" bet in roulette that pays 1:1. This bet has a $\frac{20}{38}$ probability of winning. You have enough money to place one bet on "not black." You decide to keep betting on "not black" as long as you have enough money to do so. What is the probability that you bankrupt the casino? In other words, what is the probability that you will never bust?

Solution (outline). Let $p = \frac{20}{38}$ and put $q = 1 - p$. Let $P(x)$ denote the probability that you place exactly x bets. One can show that $P(1 + 2n) = C_n q^{n+1} p^n$, for $n \geq 0$, where $C_n = \frac{\binom{2n}{n}}{n+1}$ is the n-th Catalan number. The generating function for these numbers is

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x},$$

which holds for $0 < |x| \leq 1/4$.

The probability you never bust is therefore

$$\begin{aligned} 1 - \sum_{n=0}^{\infty} C_n q^{n+1} p^n &= 1 - q \sum_{n=0}^{\infty} C_n (qp)^n \\ &= 1 - q\,\frac{1 - \sqrt{1 - 4qp}}{2qp} \\ &= 1 - \frac{q}{p} \\ &= \boxed{\frac{1}{10}}. \end{aligned}$$

Week 8. *Proposed by Matthew McMullen.*

For ten years, you put a fixed dollar amount at the end of each month into a savings account that earns an APR of 2.5% compounded monthly (before the monthly deposit). Let T represent the total amount you invested in this account. If, instead, you had made a one-time deposit of T into the same savings account, how long would it have taken you to realize the same future value?

Solution. Let D denote your monthly deposits. Then $T = 120D$. We need to solve

$$\frac{D[(1+\frac{.025}{12})^{120}-1]}{(\frac{.025}{12})} = 120D\left(1+\frac{.025}{12}\right)^{12t}$$

for t. Using mad algebra skills, we find

$$t = \frac{\ln 4 + \ln[(1+\frac{.025}{12})^{120}-1]}{12\ln(1+\frac{.025}{12})},$$

which is more than 5 years, but less than 5 years and 1 month. So it would have taken $\boxed{\text{61 months}}$ to realize the same future value.

Week 9. *Proposed by Matthew McMullen.*

Define

$$a(x,y) = \frac{2y}{x+y}\binom{x-1}{\frac{x-y}{2}},$$

where x and y are integers, $x \geq 1$, $y \geq 0$, $x \geq y$, and x and y have the same parity (i.e., are either both even or both odd). Show that $a(n,n) = 1$ for all positive integers n and that

$$a(n+1, m+1) = a(n,m) + a(n, m+2),$$

for all nonnegative integers n and m that have the same parity and satisfy $n > m$.

Solution (not really). You'll have to trust me on this one. If you expand out both sides, you will see that they're equal. The inspiration for this problem came from the Catalan numbers, which is the sequence you get as x ranges through the odd positive integers when $y = 1$ is fixed.

Week 10. *Modified Purdue U. Problem of the Week.*

Let $f(x)$ be a strictly increasing differentiable function on a bounded interval $[a,b]$. Choose c in $[a,b]$. Consider the two curvilinear triangles bounded by the vertical lines $x = a$, $x = b$, the horizontal line $y = f(c)$, and the graph of f. For which position c is the sum of the areas of these curvilinear triangles minimal?

Solution. Let $A(c)$ be the sum of the areas of the curvilinear triangles described in the question. Then

$$\begin{aligned} A(c) &= \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx \\ &= (c-a)f(c) - \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - (b-c)f(c) \\ &= (2c-a-b)f(c) - \int_a^c f(x)\,dx - \int_b^c f(x)\,dx. \end{aligned}$$

Taking the derivative with respect to c gives us

$$A'(c) = (2c-a-b)f'(c) + 2f(c) - f(c) - f(c) = (2c-a-b)f'(c).$$

Now, if $a \le c \le \frac{a+b}{2}$, then $A'(c) \le 0$, and if $\frac{a+b}{2} \le c \le b$, then $A'(c) \ge 0$ (remember that f is strictly increasing). Thus, $\boxed{c = \frac{a+b}{2}}$ minimizes A.

Week 11. *Proposed by Matthew McMullen.*

Let k be a positive integer. Find

$$\lim_{n\to\infty} \left(\frac{1}{n+k} + \frac{1}{n+k+1} + \cdots + \frac{1}{n+nk} \right).$$

Solution. Let $S(n)$ denote the above sum. By thinking of $S(n)$ as an upper and lower Riemann sum, we have

$$\int_{n+k}^{n+nk+1} \frac{1}{x}\,dx < S(n) < \int_{n+k-1}^{n+nk} \frac{1}{x}\,dx,$$

or

$$\ln\left(\frac{n+nk+1}{n+k}\right) < S(n) < \ln\left(\frac{n+nk}{n+k-1}\right).$$

By the Squeeze Theorem, $\lim_{n\to\infty} S(n) = \boxed{\ln(k+1)}$.

Week 12. *From 2015 AIME I.*

There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

Solution. We are given that $16p = k^3 - 1$ for some positive integer k. This forces k to be odd, say $k = 2a + 1$ for some $a \geq 0$. Then $8p = a(4a^2 + 6a + 3)$. Then a is even, say $a = 2b$ for some $b \geq 0$; and we have $4p = b(16b^2 + 12b + 3)$. Then b is even, say $b = 2c$ for some $c \geq 0$; and we have $2p = c(64c^2 + 24c + 3)$. Then c is even, say $c = 2d$ for some $d \geq 0$; and we have (finally!) $p = d(256d^2 + 48d + 3)$. Since p is prime, $d = 1$ and we get $p = 256 + 48 + 3 = \boxed{307}$.

Week 13. *From 2015 AIME II.*

Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

Solution. Suppose that $m = 100a + 10b + c$, where $a, b, c \in \{0, \ldots, 9\}$ and $a + b + c = 17$. Then $m = 99a + 9b + 17$. Since m is divisible by 17, and 17 and 9 are relatively prime, $11a + b$ is divisible by 17. Since $0 \leq 11a + b \leq 108$, we must have $11a + b \in \{0, 17, 34, 51, 68, 85, 102\}$. The first viable option occurs when $11a + b = 51$, whereby we see that $a = 4$, $b = 7$, and $c = 6$. So, $m = \boxed{476}$.

Week 14. *Proposed by Matthew McMullen.*

Let S be the set of all positive integers divisible by 17 whose digits sum to 17. How many integers less than 10,000 are in S?

Solution (kinda). Let $m < 10000$ be in S. Then $m = 1000a + 100b + 10c + d$, where $a, b, c, d \in \{0, \ldots, 9\}$ and $a + b + c + d = 17$. Then $m = 999a + 99b + 9c + 17$. Since m is divisible by 17, and 17 and 9 are relatively prime, $111a + 11b + c$ is divisible by 17. Reducing this mod 17 tells us that $9a + 11b + c$ is divisible by 17.

If you go through all the possible options for b, c, d when $a = 0, 1, \ldots, 9$, respectively, you will (trust me!) find 41 possible values for m. For two such examples, $m = 7361 = 17(433)$ or $m = 4913 = 17(289)$. (Can you find a less brute-force way to do this problem?)

Week 15$^+$. *Proposed by Matthew McMullen.*

Let S be the set of all positive integers divisible by 17 whose digits sum to 17. Define $a(n)$ to be the number of integers less than or equal to n that are in S. Describe the function $a(n)$. In particular, is it asymptotic to some "well-known" function?

Solution (not!). In all likelihood, this is an unsolved problem. Just for fun, the following graph of $a(n)$ for $1 \leq n \leq 1,000,000$ was generated using Mathematica.