Coffee Hour Problems and Solutions

Edited by Matthew McMullen

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Week 1. *Proposed by Matthew McMullen.*

Kaprekar's routine is when you take a number, arrange its digits in descending and then ascending order to get two numbers, and then subtract the smaller number from the bigger number. For example, Kaprekar's routine performed on the number 803 yields the number 792, since $830 - 038 = 792$. For fun, we could repeat this process on 792 to get 693, since $972 - 279 = 693$. And then we can repeat this process again on 693, and again on the resulting number, etc.

What happens when we repeat Kaprekar's routine over and over again on four-digit numbers? Try it on 2014, 2015, 2016, and 2017. Make an educated guess about what, if anything, eventually happens. Can you prove your guess for *all* four-digit numbers?

Solution. If all four digits are equal, Kaprekar's routine obviously results in 0000. It turns out that if at least two digits are different, Kaprekar's routine eventually reaches the number $\boxed{6174}$. See `http://en.wikipedia.org/wiki/6174_(number)` for more details.

Week 2. *Proposed by Matthew McMullen.*

It takes three days for a boat to travel from A to B downstream and four days to come back upstream. Assuming the velocity of the current is constant, how long will it take a wooden log to be carried from A to B by the current?

Solution. Let x be the speed of the boat in still water, in miles per day, and let y be the speed of the current, in miles per day. Then the distance the boat travels downstream is $3(x + y)$, and the distance the boat travels upstream is $4(x - y)$. Equating these yields $x = 7y$; so the distance from A to B is $3(7y + y) = 24y$. The log will be carried at a rate of y miles per day, so it will take $\frac{24y}{y} = \boxed{24}$ days for the log to go from A to B.

Week 3. *Proposed by Matthew McMullen.*

Let N be any number with two or more digits such that the digits are non-decreasing and the tens digit is strictly less than the ones digit. Show that the digits of $9N$ sum to 9. For example, if $N = 22379$, then $9N = 201411$, and $2+0+1+4+1+1 = 9$.

Solution. (Every number is written in base 10 notation.) Suppose $N = c_n c_{n-1} \dots c_2 c_1$. For each i, put $9c_i = a_i b_i$; clearly, $a_i + b_i = 9$. Then $9N = a_1 b_1 + a_2 b_2 0 + a_3 b_3 00 + \dots$. Since the digits of N are non-decreasing with the tens digit strictly less than the ones digit, there will be a carry in all but the ones position. Thus, the sum of the digits of $9N$ is given by

$$\begin{aligned} b_1 + (a_1 + b_2 - 10) + (a_2 + b_3 + 1 - 10) + (a_3 + b_4 + 1 - 10) + \cdots \\ + (a_{n-1} + b_n + 1 - 10) + (a_n + 1), \end{aligned}$$

which rearranges to

$$(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) - 9(n-1) = 9n - 9(n-1) = 9.$$

Week 4. *Proposed by Matthew McMullen.*

Show that there is some multiple of 2017 that consists solely of ones.

Solution. Consider the numbers $1, 11, 111, \dots, 11\cdots11$, where the last number has 2018 digits. Since there are only 2017 possible remainders when dividing by 2017, two of the numbers in our list must leave the same remainder upon division by 2017. If we subtract these two numbers, we get a number of the form $11\cdots1100\cdots00$ that is divisible by 2017. In other words, 2017 divides a number of the form $N \times 10^k$, for some $N = 11\cdots11$ and some nonnegative k. Since the gcd of 2017 and 10^k is 1, 2017 must divide N.

Week 5. *Proposed by Matthew McMullen.*

Find all pairs of integers a, b such that the vectors $\langle a, 2, 3\rangle$, $\langle 4, b, 6\rangle$, and $\langle 7, 8, 9\rangle$ are linearly dependent over $\mathbf{R^3}$.

Solution. Let

$$M = \begin{pmatrix} a & 2 & 3 \\ 4 & b & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then we have $\det M = a(9b-48)-2(36-42)+3(32-7b) = 9ab-48a-21b+108$, and we want to find all integers a, b such that $\det M = 0$. Solving for a gives us $a = \frac{7b-36}{3b-16}$. If $b > 6$ or $b < 4$, then $2 < a < 3$; so our only possible values for b are 4, 5, and 6. The respective values of a are 2, 1, and 3. Thus, there are three (a, b) pairs: $\boxed{(1,5),\ (2,4),\text{ and }(3,6)}$.

Week 6. *Problem A1 from the 1998 Putnam Exam.*

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Solution. Let s be the side-length of the cube. If you slice the original cone parallel to the base at the top of the cube, you get a smaller cone of diameter $s\sqrt{2}$ (the length of the diagonal of each face of the cube) and height $3-s$. The 2 to 3 ratio of the base diameter to the height is preserved, so we have $\frac{2}{3} = \frac{s\sqrt{2}}{3-s}$, or $s = \boxed{\frac{6}{3\sqrt{2}+2}}$.

Week 7. *Problem A1 from the 1996 Putnam Exam.*

Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume the sides of the square will be parallel to the sides of the rectangle.

Solution (curtesy of student Jeffrey Guillott). Suppose you have two squares, one of area x and the other of area y, where $x + y = 1$; and, without loss of generality, $x \geq y$. The most efficient way to pack these squares into a rectangle is to place them side by side with their bases lined up. This rectangle will have area $\sqrt{x}(\sqrt{x} + \sqrt{y})$, or $x + \sqrt{x}\sqrt{1-x}$.

Therefore, we need to maximize the function

$$f(x) = x + \sqrt{x - x^2}$$

on the interval $[1/2, 1]$. Setting the derivative to zero and solving for x yields the only critical point, $x_0 = \frac{2+\sqrt{2}}{4}$. Then A is the maximum of $f(1/2) = 1$, $f(1) = 1$, and $f(x_0) = \frac{1+\sqrt{2}}{2}$. Thus, $A = \boxed{\frac{1+\sqrt{2}}{2}}$.

Week 8. *Modified from a Purdue University Problem of the Week.*

A cube is inscribed in the unit sphere $x^2 + y^2 + z^2 = 1$. Let A, B, C, and D denote the vertices of one face of the cube. Let O denote the center of the sphere, and let P denote a point on the sphere. Find

$$\cos^2(POA) + \cos^2(POB) + \cos^2(POC) + \cos^2(POD).$$

Solution. Without loss of generality, let's choose the "front" face of the cube, those points with positive x-coordinate. Then one possible ordering gives $A = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $B = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $C = (1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$, and $D = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$.

Suppose $P = (x, y, z)$. Using either the Law of Cosines, or the dot product (which amounts to the same thing), one can show that

$$\begin{aligned}
\cos(POA) &= \frac{x+y+z}{\sqrt{3}}, \\
\cos(POB) &= \frac{x-y+z}{\sqrt{3}}, \\
\cos(POC) &= \frac{x-y-z}{\sqrt{3}}, \text{ and} \\
\cos(POD) &= \frac{x+y-z}{\sqrt{3}}.
\end{aligned}$$

Then, after multiplying out (using the fact that $x^2 + y^2 + z^2 = 1$) we see that

$$\cos^2(POA) + \cos^2(POB) + \cos^2(POC) + \cos^2(POD) = \boxed{\frac{4}{3}}.$$

Week 9. *Proposed by Matthew McMullen.*

Find $a > 0$ such that

$$\int_0^\infty \frac{1}{x^{1-a} + x^{1+a}}\, dx = 2014.$$

Solution. Multiplying top and bottom by x^{a-1} and then using the substitution $u = x^a$ gives us

$$\begin{aligned}
\int_0^\infty \frac{1}{x^{1-a}+x^{1+a}}\,dx &= \int_0^\infty \frac{x^{a-1}}{1+x^{2a}}\,dx \\
&= \frac{1}{a}\int_0^\infty \frac{1}{1+u^2}\,du \\
&= \frac{1}{a}\tan^{-1} u\Big|_{u=0}^{u\to\infty} \\
&= \frac{\pi}{2a}.
\end{aligned}$$

Solving $\frac{\pi}{2a} = 2014$ gives us $a = \boxed{\frac{\pi}{4028}}$.

Week 10. *Proposed by Matthew McMullen.*

Let $\epsilon > 0$. Find sequences a_n and b_n such that $\sum_{n=1}^\infty a_n$ converges, $\sum_{n=1}^\infty b_n$ diverges, and there exists some positive integer n_0 with $\frac{1}{n^{1+\epsilon}} < a_n, b_n < \frac{1}{n}$, for all $n \geq n_0$.

Solution. Let $a_n = \frac{1}{n^{1+\epsilon/2}}$ and $b_n = \frac{1}{(n+1)\ln(n+1)}$. Then $\sum_{n=1}^\infty a_n$ converges and $\sum_{n=1}^\infty b_n$ diverges (both by, say, the Integral Test). Since $1+\epsilon > 1+\epsilon/2 > 1$, $\frac{1}{n^{1+\epsilon}} < a_n < \frac{1}{n}$, for all $n \geq 1$. Also, since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\ln(n+1) = \infty \text{ and } \lim_{n\to\infty}\frac{(1+1/n)\ln(n+1)}{n^\epsilon} = 0,$$

$(n+1)\ln(n+1)$ is eventually greater than n and less than $n^{1+\epsilon}$; in other words, there exists some positive integer n_0 with $\frac{1}{n^{1+\epsilon}} < b_n < \frac{1}{n}$, for all $n \geq n_0$.

Week 11. *From our calculus textbook.*

Find the value of a for which the limit

$$\lim_{x\to 0}\frac{\sin(ax) - \sin x - x}{x^3}$$

is finite, and evaluate the limit.

Solution. If $a = 2$, we can apply L'Hospital's Rule three times and see that the limit is $-\frac{7}{6}$. Alternatively, we can use Taylor polynomials to see that

$$\frac{\sin(ax) - \sin x - x}{x^3} = \frac{a-2}{x^2} + \frac{1-a^3}{6} + f(x),$$

where $f(x) \to 0$ as $x \to 0$. Thus, $a = 2$ and the limit is $\boxed{-\frac{7}{6}}$.

Week 12. *Proposed by Matthew McMullen.*

It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - x^2} = \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

for all non-integers x. Use this identity to find

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9n^2 - 1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. (The above identity also holds as $x \to 0$.)

Solution. For (a)–(c), notice that

$$\sum_{n=1}^{\infty} \frac{1}{k^2 n^2 - 1} = \frac{1}{k^2} \sum_{n=1}^{\infty} \frac{1}{n^2 - (1/k)^2} = \frac{1 - \frac{\pi}{k} \cot \frac{\pi}{k}}{2}.$$

Using this identity, we see that the answer to (a) is $\frac{1}{2}$, the answer to (b) is $\frac{9 - \pi\sqrt{3}}{18}$, and the answer to (c) is $\frac{4-\pi}{8}$.

For (d), one can use L'Hospital's Rule, or Taylor series, to show that

$$\lim_{x \to 0} \frac{1 - \pi x \cot(\pi x)}{2x^2} = \frac{\pi^2}{6}.$$

Week 13. *Proposed by Matthew McMullen.*

Show that the surface area of the part of a sphere trapped between two parallel planes depends only on the distance between the planes.

Solution. Without loss of generality, suppose that the sphere is generated by rotating the top half of the circle $x^2+y^2=r^2$ about the x-axis and that the two planes are $x=a$ and $x=b$, where $-r \le a < b \le r$. By implicit differentiation, $2x+2yy'=0$, or $yy'=-x$. Then the surface area we want is given by

$$\begin{aligned}\int_a^b 2\pi y\sqrt{1+(y')^2}\,dx &= 2\pi\int_a^b \sqrt{y^2+(yy')^2}\,dx\\ &= 2\pi\int_a^b \sqrt{y^2+x^2}\,dx\\ &= 2\pi\int_a^b r\,dx\\ &= 2\pi r(b-a).\end{aligned}$$

Week 14. *Proposed by Matthew McMullen.*

Find all real numbers x such that

$$x=\sum_{n=1}^{\infty}\frac{n(n+1)}{x^n}.$$

Solution. By the root test, the right-hand side of the given equation only makes sense if $|x|>1$. Let $u=1/x$. Then we need to solve

$$\frac{1}{u^2}=\sum_{n=1}^{\infty}n(n+1)u^{n-1},$$

for $|u|<1$.

For $|u|<1$,

$$\sum_{n=1}^{\infty}u^{n+1}=\frac{u^2}{1-u},$$

since this is a convergent geometric series. Taking the second derivative of both sides with respect to u gives us

$$\sum_{n=1}^{\infty}n(n+1)u^{n-1}=\frac{2}{(1-u)^3}.$$

We therefore need to solve

$$\frac{1}{u^2} = \frac{2}{(1-u)^3},$$

or

$$x^2 = \frac{2}{(1-1/x)^3}.$$

Multiplying the last equation out shows us that x is the (unique) real solution to the equation $x^3 - 3x^2 + x - 1 = 0$, or $x \approx 2.76929$.