Coffee Hour Problems

Edited by Matthew McMullen

Spring 2015

Week 1. *Proposed by Matthew McMullen.*

Show that
$$\arccos\left(\frac{2x}{x^2+1}\right) = \arctan x - \arctan\frac{1}{x}$$
if and only if $x \geq 1$.

Week 2. *Proposed by Matthew McMullen.*

Suppose a_0, a_1, a_2, a_3, and a_4 are all non-negative and satisfy
$$\sum_{n=0}^{4} a_n = 1,\ \sum_{n=0}^{4} n a_n = 1,\ \sum_{n=0}^{4} n^2 a_n = 2,\ \text{ and } \sum_{n=0}^{4} n^3 a_n = 5.$$
What is the largest possible value of a_0?

Week 3. *Proposed by Matthew McMullen.*

Evaluate
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\cot^3 x}\,dx.$$
(Hint: Try the substitution $u = \frac{\pi}{2} - x$.)

Week 4. *Proposed by Matthew McMullen.*

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 2016 \\ 4y + 5z &= 2014 \end{aligned}$$

given that x, y, and z are integers; $z > 0$; and xyz is as small as possible.

Week 5. *Proposed by Matthew McMullen.*

The game of craps involves a shooter repeatedly rolling two fair dice. The simplest bet in this game is the "pass" bet. If the shooter's first roll is a 7 or 11, "pass" wins. If the shooter's first roll is a 2, 3, or 12, "pass" loses. Otherwise, whatever number is rolled first must be rolled again before a 7 is rolled in order for "pass" to win. Wikipedia says that the chance of winning a "pass" bet is $\frac{244}{495}$. Prove that this is correct.

Week 6. *Proposed by Matthew McMullen.*

You have enough money to place one bet on "pass" in craps. Recall from last week that the probability of winning this bet is $\frac{244}{495}$. The casino pays 1:1 for this bet. You decide to keep betting on "pass" as long as you have enough money to do so. Find the probability that you will be able to place more than nine total bets.

Week 7. *Proposed by Matthew McMullen.*

Suppose that a casino has a "not black" bet in roulette that pays 1:1. This bet has a $\frac{20}{38}$ probability of winning. You have enough money to place one bet on "not black." You decide to keep betting on "not black" as long as you have enough money to do so. What is the probability that you bankrupt the casino? In other words, what is the probability that you will never bust?

Week 8. *Proposed by Matthew McMullen.*

For ten years, you put a fixed dollar amount at the end of each month into a savings account that earns an APR of 2.5% compounded monthly (before the monthly deposit). Let T represent the total amount you invested in this account. If, instead, you had made a one-time deposit of T into the same savings account, how long would it have taken you to realize the same future value?

Week 9. *Proposed by Matthew McMullen.*

Define

$$a(x,y) = \frac{2y}{x+y}\binom{x-1}{\frac{x-y}{2}},$$

where x and y are integers, $x \geq 1$, $y \geq 0$, $x \geq y$, and x and y have the same parity (i.e., are either both even or both odd). Show that $a(n,n) = 1$ for all positive integers n and that

$$a(n+1, m+1) = a(n,m) + a(n, m+2),$$

for all nonnegative integers n and m that have the same parity and satisfy $n > m$.

Week 10. *Modified Purdue U. Problem of the Week.*

Let $f(x)$ be a strictly increasing differentiable function on a bounded interval $[a,b]$. Choose c in $[a,b]$. Consider the two curvilinear triangles bounded by the vertical lines $x = a$, $x = b$, the horizontal line $y = f(c)$, and the graph of f. For which position c is the sum of the areas of these curvilinear triangles minimal?

Week 11. *Proposed by Matthew McMullen.*

Let k be a positive integer. Find

$$\lim_{n\to\infty}\left(\frac{1}{n+k} + \frac{1}{n+k+1} + \cdots + \frac{1}{n+nk}\right).$$

Week 12. *From 2015 AIME I.*

There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

Week 13. *From 2015 AIME II.*

Let m be the least positive integer divisible by 17 whose digits sum to 17. Find m.

Week 14. *Proposed by Matthew McMullen.*

Let S be the set of all positive integers divisible by 17 whose digits sum to 17. How many integers less than 10,000 are in S?

Week 15$^+$. *Proposed by Matthew McMullen.*

Let S be the set of all positive integers divisible by 17 whose digits sum to 17. Define $a(n)$ to be the number of integers less than or equal to n that are in S. Describe the function $a(n)$. In particular, is it asymptotic to some "well-known" function?