Coffee Hour Problems

Edited by Matthew McMullen

Fall 2014

Week 1. *Proposed by Matthew McMullen.*

Kaprekar's routine is when you take a number, arrange its digits in descending and then ascending order to get two numbers, and then subtract the smaller number from the bigger number. For example, Kaprekar's routine performed on the number 803 yields the number 792, since $830 - 038 = 792$. For fun, we could repeat this process on 792 to get 693, since $972 - 279 = 693$. And then we can repeat this process again on 693, and again on the resulting number, etc.

What happens when we repeat Kaprekar's routine over and over again on four-digit numbers? Try it on 2014, 2015, 2016, and 2017. Make an educated guess about what, if anything, eventually happens. Can you prove your guess for *all* four-digit numbers?

Week 2. *Proposed by Matthew McMullen.*

It takes three days for a boat to travel from A to B downstream and four days to come back upstream. Assuming the velocity of the current is constant, how long will it take a wooden log to be carried from A to B by the current?

Week 3. *Proposed by Matthew McMullen.*

Let N be any number with two or more digits such that the digits are non-decreasing and the tens digit is strictly less than the ones digit. Show that the digits of $9N$ sum to 9. For example, if $N = 22379$, then $9N = 201411$, and $2 + 0 + 1 + 4 + 1 + 1 = 9$.

Week 4. *Proposed by Matthew McMullen.*

Show that there is some multiple of 2017 that consists solely of ones.

Week 5. *Proposed by Matthew McMullen.*

Find all pairs of integers a, b such that the vectors $\langle a, 2, 3\rangle$, $\langle 4, b, 6\rangle$, and $\langle 7, 8, 9\rangle$ are linearly dependent over $\mathbf{R^3}$.

Week 6. *Problem A1 from the 1998 Putnam Exam.*

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Week 7. *Problem A1 from the 1996 Putnam Exam.*

Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume the sides of the square will be parallel to the sides of the rectangle.

Week 8. *Modified from a Purdue University Problem of the Week.*

A cube is inscribed in the unit sphere $x^2 + y^2 + z^2 = 1$. Let A, B, C, and D denote the vertices of one face of the cube. Let O denote the center of the sphere, and let P denote a point on the sphere. Find

$$\cos^2(POA) + \cos^2(POB) + \cos^2(POC) + \cos^2(POD).$$

Week 9. *Proposed by Matthew McMullen.*

Find $a > 0$ such that

$$\int_0^\infty \frac{1}{x^{1-a} + x^{1+a}}\,dx = 2014.$$

Week 10. *Proposed by Matthew McMullen.*

Let $\epsilon > 0$. Find sequences a_n and b_n such that $\sum_{n=1}^\infty a_n$ converges, $\sum_{n=1}^\infty b_n$ diverges, and there exists some positive integer n_0 with $\frac{1}{n^{1+\epsilon}} < a_n, b_n < \frac{1}{n}$, for all $n \geq n_0$.

Week 11. *From our calculus textbook.*

Find the value of a for which the limit

$$\lim_{x\to 0} \frac{\sin(ax) - \sin x - x}{x^3}$$

is finite, and evaluate the limit.

Week 12. *Proposed by Matthew McMullen.*

It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - x^2} = \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

for all non-integers x. Use this identity to find

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9n^2 - 1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. (The above identity also holds as $x \to 0$.)

Week 13. *Proposed by Matthew McMullen.*

Show that the surface area of the part of a sphere trapped between two parallel planes depends only on the distance between the planes.

Week 14. *Proposed by Matthew McMullen.*

Find all real numbers x such that

$$x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n}.$$