Coffee Hour Problems and Solutions

Edited by Matthew McMullen

Spring 2014

Week 1. *Proposed by Matthew McMullen.*

Find the last two digits of $89^{2014} + 1$.

Solution. Notice that the last two digits of successive powers of 11 (starting with 11^1) are 11, 21, 31, 41, 51, 61, 71, 81, 91, 01, ... Therefore, working mod 100, we have

$$89^{2014} + 1 \equiv (-11)^{2014} + 1 \equiv (11)^4 + 1 \equiv 41 + 1 = \boxed{42}.$$

Week 2. *Proposed by Matthew McMullen.*

Four consecutive even integers are removed from the set $\{1, 2, 3, \ldots n\}$. The average of the remaining numbers is 51.5625. Find the value of n and the values of the integers that were removed.

Solution. Let k, $k+2$, $k+4$, and $k+6$ be the integers that are removed from the set. Then we must have

$$\frac{\frac{n(n+1)}{2} - (4k+12)}{n-4} = 51.5625 = \frac{825}{16},$$

where k is even and $2 \leq k \leq n-6$. This is equivalent to

$$8n^2 - 64k + 3108 = 817n. \tag{1}$$

Since 4 divides the left-hand side of (1) and 4 does not divide 817, we must have that 4 divides n; say $n = 4a$, for some $a \geq 2$.

Plugging this into (1) and dividing by 4 yields

$$32a^2 - 817a = 16k - 777. \tag{2}$$

Reducing (2) mod 16 tells us that a is 9 more than a multiple of 16; say $a = 16b + 9$, for some $b \geq 0$.

Plugging this into (2) and dividing by 16 yields $k = 512b^2 - 241b - 249$. Also, we have $n = 4a = 4(16b + 9)$. The only value of b that works with our restrictions on k and n is $b = 1$. Thus, $\boxed{n = 100}$ and the numbers removed were $\boxed{\text{22, 24, 26, and 28}}$.

Week 3. *Proposed by Matthew McMullen.*

Find all pairs of non-negative integers x, y such that $y^2 = x^3 - 3x + 2$.

Solution. Factoring the right-hand side gives us $y^2 = (x-1)^2(x+2)$; so $y = |x-1|\sqrt{x+2}$. Thus, either $x = 1$ or $x + 2 = k^2$ for some integer $k \geq 2$. The solution set is $\boxed{\{(1,0)\} \cup \{(k^2-2, k^3-3k) \mid k \geq 2\}}$.

Week 4. *Proposed by Matthew McMullen.*

Find an explicit continuous function, $f(x)$, such that
(i) f is differentiable for all $x \neq 0$,
(ii) $f(0) = 1/2$,
(iii) $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 0$, and
(iv) $\lim_{x\to 0} \frac{f(x)-1/2}{x} = \infty$.

Solution. A shift of the arctangent function would give us Property (iii), and a shift of the cube root function would give us Property (iv), so my idea was to compose these two functions. For example,

$$\boxed{f(x) = \tfrac{1}{\pi}\arctan(\sqrt[3]{x}) + \tfrac{1}{2}}$$

is one possible answer. Another possibility is

$$\boxed{f(x) = \sqrt[3]{\tfrac{\arctan x}{4\pi}} + \tfrac{1}{2}}.$$

Week 5. *Proposed by Matthew McMullen.*

Use the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ to find

$$\int_0^1 \int_0^1 \ln(1 - xy)\, dy\, dx.$$

Solution (outline). We will use the fact that $\ln(1-u) = -\sum_{n=1}^{\infty} \frac{u^n}{n}$ for $|u| < 1$ and work informally (ignoring questions about convergence and interchanging infinite sums and integrals). We have

$$\begin{aligned}
\int_0^1 \int_0^1 \ln(1 - xy)\, dy\, dx &= -\sum_{n=1}^{\infty} \int_0^1 \int_0^1 \frac{x^n y^n}{n}\, dy\, dx \\
&= -\sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n(n+1)}\, dx \\
&= -\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right) + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\
&= -1 + \frac{\pi^2}{6} - 1 \\
&= \boxed{\tfrac{\pi^2}{6} - 2}.
\end{aligned}$$

Week 6. *Proposed by Matthew McMullen.*

Let $f(x) = 1 + a_1 \cos(2\pi x) + a_2 \cos(4\pi x)$, where a_1 and a_2 are constants such that $f(x) \geq 0$ for all x. Find the largest possible value of $f(0)$.

Solution. Notice that $f(0) = 1 + a_1 + a_2$. Since $0 \leq f(1/3) = 1 - a_1/2 - a_2/2$, the best we can hope for is $a_1 + a_2 = 2$. To see if this is possible let $a_2 = 2 - a_1$, and differentiate to get (after some simplifying and factoring)

$$f'(x) = -2\pi \sin(2\pi x)[a_1 + 4(2 - a_1)\cos(2\pi x)].$$

Our critical points in $[0, 1)$ are $x = 0$, $x = 1/2$, and all x such that

$$\cos(2\pi x) = \frac{-a_1}{4(2 - a_1)}. \tag{3}$$

Notice that $f(0) = 3$ and $f(1/2) = 3 - 2a_1$. If x_0 satisfies (3), we have

$$f(x_0) = -\frac{(3a_1 - 4)^2}{8(2 - a_1)}.$$

The only way this can be non-negative is if $a_1 = 4/3$. Thus, $a_2 = 2/3$ and the largest possible value of $f(0)$ is indeed $\boxed{3}$.

Week 7. *Proposed by Matthew McMullen.*

The vertices of a polygon are $(-1, 0)$, $(1, 0)$, (x_1, y_1), and (x_2, y_2), where the last two points are on the top half of the unit circle. Where should these points be placed to maximize the perimeter of the polygon?

Solution. Intuitively, it seems that the points should be placed at equally-spaced spots on the top half of the unit circle, specifically at the points $(\pm 1/2, \sqrt{3}/2)$. To prove this is actually the case, let α_1 be the angle formed by $(1, 0)$, $(0, 0)$, and (x_1, y_1); let α_2 be the angle formed by (x_1, y_1), $(0, 0)$, and (x_2, y_2); and let α_3 be the angle formed by (x_2, y_2), $(0, 0)$, and $(-1, 0)$. Then, using the Law of Cosines and the half-angle formula, we find that the perimeter of the resulting polygon is

$$2 + 2\left(\sin\left(\frac{\alpha_1}{2}\right) + \sin\left(\frac{\alpha_2}{2}\right) + \sin\left(\frac{\alpha_3}{2}\right)\right).$$

For angles $0 \le A, B, C \le \pi$, it can be shown (using Jensen's inequality, for example) that

$$\frac{\sin A + \sin B + \sin C}{3} \le \sin\left(\frac{A + B + C}{3}\right),$$

with equality iff $A = B = C$. Therefore, the above perimeter is maximized when $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$, which is what we set out to prove.

Week 8. *Proposed by Matthew McMullen.*

You are standing at the center of the $[-1, 1] \times [-1, 1]$ square.

(a) If you walk off in a random direction, what is the average, or expected, distance you will walk until you hit the edge of the square?

(b) If a point is chosen randomly on the square's edge, what is the average distance from you to the chosen point?

Solution. (a) Let θ represent the direction that you walk in. Due to symmetry, it is enough to consider $0 \le \theta \le \frac{\pi}{4}$. Then the distance to the edge of the square is given by $\sec\theta$, and the average distance is

$$\frac{4}{\pi}\int_0^{\pi/4} \sec\theta \, d\theta = \frac{4}{\pi}\ln(\sqrt{2}+1).$$

(b) The distance from the origin to the point $(1, y)$ on the square's edge is given by $\sqrt{y^2+1}$. Again, by symmetry, it is enough to consider $0 \le y \le 1$. In this case, the average distance is given by

$$\int_0^1 \sqrt{y^2+1}\, dy = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln(\sqrt{2}+1).$$

Week 9. *Purdue University Problem of the Week.*

Does the series

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2n}$$

converge?

Solution. Notice that $a_4 = \frac{35}{128} > \frac{1}{4}$; and, if $a_k > \frac{1}{k}$, then

$$a_{k+1} = a_k \cdot \frac{2k+1}{2k+2} > \frac{1}{k}\cdot\frac{2k+1}{2(k+1)} = \frac{1}{k+1}\left(1+\frac{1}{2k}\right) > \frac{1}{k+1}.$$

This shows, by induction, that $a_n > \frac{1}{n}$ for $n \ge 4$. Thus, by comparing to the harmonic series, the given series $\boxed{\text{diverges}}$.

Week 10. *Proposed by Matthew McMullen.*

A delivery company will only accept a box for shipment if the sum of its length and girth (distance around) does not exceed some maximum length, M. You would like to ship a box with regular n-gon ends that has the largest possible volume. Show that the length of this box is $M/3$ (regardless of n).

Solution. Let s be the side-length of the n-gon, and let l be the length of the box. We are given that $l + ns \le M$, or $l \le M - ns$, and we wish to maximize the volume of the box. It can be shown that the area of an n-gon with side-length s is given by $\frac{n}{4}\cot(\frac{\pi}{n})s^2$. Therefore, we wish to maximize $\frac{n}{4}\cot(\frac{\pi}{n})s^2 l$, under the constraint that $l \le M - ns$. In other words, we want to maximize $C_n(Ms^2 - ns^3)$, where C_n is a constant (depending on n).

Using calculus, this maximum is easily seen to occur when $s = \frac{2M}{3n}$. The corresponding length is $l = M - n \cdot \frac{2M}{3n} = \frac{M}{3}$, as desired.

Week 11. *Purdue University Problem of the Week.*

Let f be a positive and continuous function on the real line which satisfies $f(x+1) = f(x)$ for all numbers x. Prove

$$\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})} dx \ge 1.$$

Solution. Put $g(x) = \frac{f(x)}{f(x+\frac{1}{2})}$. Since f is positive and continuous on the real line, g is also positive and continuous on the real line. In particular, g is integrable on any bounded interval. Notice that $g(x+\frac{1}{2}) = \frac{f(x+\frac{1}{2})}{f(x+1)} = \frac{f(x+\frac{1}{2})}{f(x)} = \frac{1}{g(x)}$ for all x. Also, since g is positive and $(g(x)-1)^2 \ge 0$ for all x, we have $g(x) + \frac{1}{g(x)} \ge 2$ for all x.

Therefore,

$$\begin{aligned}
\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\, dx &= \int_0^{1/2} g(x)\, dx + \int_{1/2}^1 g(x)\, dx \\
&= \int_0^{1/2} g(x)\, dx + \int_0^{1/2} g\big(x + \frac{1}{2}\big)\, dx \\
&= \int_0^{1/2} \left(g(x) + \frac{1}{g(x)} \right) dx \\
&\ge \frac{1}{2} \cdot 2 \\
&= 1.
\end{aligned}$$

Week 12. *Proposed by Matthew McMullen.*

A one-meter length of wire is cut into three pieces. The first piece is formed into an equilateral triangle, the second piece is formed into a square, and the third piece is formed into a circle. How should the wire be cut to minimize the total area enclosed by these three pieces?

Solution (outline). Let x, y, and z be the lengths that are formed into the triangle, square, and circle, respectively. Then we wish to minimize $\frac{\sqrt{3}x^2}{36} + \frac{y^2}{16} + \frac{z^2}{4\pi}$ for $x, y, z \geq 0$ and $x + y + z = 1$. Put $a = \frac{1}{2\pi+8+6\sqrt{3}}$. Using Lagrange multipliers, it can be shown that the total area is minimized when

$$\begin{aligned} x &= 6\sqrt{3}a \approx 0.42116\,\text{m}, \\ y &= 8a \approx 0.32421\,\text{m, and} \\ z &= 2\pi a \approx 0.25463\,\text{m}. \end{aligned}$$

Week 13. *From the 2014 AIME I.*

The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

Solution. We are given that $N^2 \equiv N \pmod{10000}$. In other words, $10000 = 2^4 \cdot 5^4$ divides $N^2 - N = N(N-1)$. Since the digit a is not zero and N and $N-1$ are co-prime, we must have either $2^4 = 16$ divides N and $5^4 = 625$ divides $N-1$, or vice versa.

In the later case, we have $n = 625k = 16l + 1$ for some positive integers k and l. Working mod 16, we see that k is one more than a multiple of 16. Then the last four digits of n are the last four digits of $625(17) = 10625$, contradicting the fact that a is nonzero.

Therefore, $n = 16k = 625l + 1$ for some positive integers k and l. Working mod 16, we see that l is one less than a multiple of 16. Then the last four digits of n are the last four digits of $625(15) + 1 = 9376$. Thus, the three-digit number abc is $\boxed{937}$.

Week 14. *From the 2014 AIME II.*

The repeating decimals $0.abab\overline{ab}$ and $0.abcabc\overline{abc}$ satisfy

$$0.abab\overline{ab} + 0.abcabc\overline{abc} = \frac{33}{37},$$

where a, b, and c are (not necessarily distinct) digits. Find the three-digit number abc.

Solution. Put $N = 0.abab\overline{ab}$ and $M = 0.abcabc\overline{abc}$. Then $N = \frac{10a+b}{99}$ and $M = \frac{100a+10b+c}{999}$. We are given that $N + M = \frac{33}{37}$, and clearing fractions yields

$$3 \cdot 37(10a + b) + 11(100a + 10b + c) = 3^4 11^2.$$

Thus, 11 divides $10a + b$, which can only happen if $a = b$. Then the above equation reduces to $221a + c = 891$, and we see that $a = b = 4$ and $c = 7$, giving us an answer of $\boxed{447}$.

Week 15. *(Re)Proposed by Matthew McMullen.*

Is it possible for the sum of two rational numbers to equal the product of their reciprocals? (This question was published some time ago with an incorrect solution!)

Solution. It is not possible, but I don't have an elementary proof of this. Suppose two such rationals did exist, say $r + s = \frac{1}{rs}$. Then $r^2 + sr - 1/s$ must have a rational discriminant. Therefore, $s^2 + 4/s = t^2$, for some rational number t. Multiplying through by $16/s^2$ yields

$$16 + \left(\frac{4}{s}\right)^3 = \left(\frac{4t}{s}\right)^2.$$

This gives us a rational solution to the equation $y^2 = x^3 + 16$, where $x \neq 0$. Using the theory of elliptic curves[1], it can be shown, however, that the only rational solutions to this equation are the points $(0, \pm 4)$.

[1]...and here's where it gets ugly!