Coffee Hour Problems and Solutions

Edited by Matthew McMullen

Fall 2013

Week 1. *Proposed by Matthew McMullen.*

Find all pairs of positive integers x and y such that $20x + 17y = 2017$.

Solution. By inspection, $x = 100$ and $y = 1$ is one such pair of positive integers. This suggests that we should make the substitution $x = 100 - u$ and $y = 1 + v$, where u and v are nonnegative integers. Then our equation becomes $20u = 17v$. Since 20 and 17 are relatively prime, u is a multiple of 17. Also, since x must be a positive integer, the possible values of u are 0, 17, 34, 51, 68, and 85. Therefore, there are 6 possible ordered pairs (x, y): $(100, 1)$, $(83, 21)$, $(66, 41)$, $(49, 61)$, $(32, 81)$, and $(15, 101)$.

Week 2. *Proposed by Jeremy Moore.*

Find all pairs of positive integers a and b, with $a < b$, such that 4 is the largest integer that divides both a and b and 480 is the smallest positive integer that is divisible by both a and b. (In other words, $\gcd(a, b) = 4$ and $\operatorname{lcm}(a, b) = 480$.)

Solution. Since $\gcd(a, b) = 4$ and $a < b$, there exists positive integers n and k such that $a = 4n$, $b = 4k$, $\gcd(n, k) = 1$, and $n < k$. By the well-known fact that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$, we have $nk = 120$. There are four pairs (n, k) that satisfy all of the requirements: $(1, 120)$, $(3, 40)$, $(5, 24)$, and $(8, 15)$. The corresponding pairs (a, b) are $\boxed{(4, 480), (12, 160), (20, 96), \text{ and } (32, 60)}$.

Week 3. *Proposed by Matthew McMullen.*

Find, with proof, the last two digits of 2^{2013}.

Solution. Searching for a pattern in the last two digits of successive powers of two, we see that 2^2 and 2^{22} both end in 04. Since $2013 = 13 + 20(100)$, this means that the last two digits of 2^{2013} are the same as the last two digits of 2^{13}, which are $\boxed{92}$.

Week 4. *Proposed by Matthew McMullen.*

An online quiz consists of five multiple-choice questions, each with four possible choices. You can take the quiz three times (with different questions each time), and your best single score becomes your final grade for the quiz. You haven't studied at all, so you randomly guess the answers each of the three times you take the quiz. What is the probability that your final grade for the quiz is 20%?

Solution 1. Let $p(x)$ be the probability of getting a final grade of $(20x)\%$ or less for the quiz. The only way to score a 0% is to guess incorrectly for all fifteen total questions. Thus, $p(0) = (3/4)^{15}$. To score a 0% or a 20% overall, you need to score a 0% or a 20% on all three attempts individually. Thus,

$$p(1) = \left(\left(\frac{3}{5}\right)^5 + 5\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4\right)^3 = \left(\frac{81}{128}\right)^3.$$

To find the probability of getting exactly 20%, simply subtract these two probabilities: $(81/128)^3 - (3/4)^{15} = 257748885/1073741824 \approx \boxed{0.2400}$.

Solution 2. Let $p(x, y, z)$ be the probability of getting exactly x correct on the first attempt, y correct on the second attempt, and z correct on the third attempt. Then $p(1, 1, 1) = (5(1/4)(3/4)^4)^3$. Also, $p(1, 1, 0) = p(1, 0, 1) = p(0, 1, 1) = 5(1/4)(3/4)^4 \cdot 5(1/4)(3/4)^4 \cdot (3/4)^5$ and $p(1, 0, 0) = p(0, 1, 0) = p(0, 0, 1) = 5(1/4)(3/4)^4 \cdot (3/4)^5 \cdot (3/4)^5$. Adding the probabilities of all seven possible ways of scoring exactly 20% on the quiz gives us the above answer.

Week 5. *Proposed by Matthew McMullen.*

If you look at a table of critical values of the Student's t-distribution, you will find that, with one degree of freedom, the value of t that has area 0.025 to its right is approximately 12.706. With two degrees of freedom, this value is approximately 4.303. What are the exact values of these critical numbers?

Solution. Let $f_v(x)$ denote the probability density function of the Student's t-distribution with v degrees of freedom. Then

$$f_v(x) = \frac{C_v}{(1 + x^2/v)^{(v+1)/2}},$$

where the constant C_v is determined by the fact that $\int_{-\infty}^{\infty} f_v(x)\, dx = 1$.

Using the substitution $\tan u = x/\sqrt{v}$, we obtain

$$\int f_v(x)\, dx = C_v\sqrt{v} \int \cos^{v-1} u\, du = \begin{cases} C_1 \arctan x + C & \text{if } v = 1 \\ C_2\sqrt{2}\frac{x}{\sqrt{x^2+2}} + C & \text{if } v = 2 \end{cases}.$$

Thus, $C_1 = 1/\pi$ and $C_2 = 1/(2\sqrt{2})$, and we need to find t_1 and t_2 such that

$$0.025 = \int_{t_1}^{\infty} f_1(x)\, dx = \frac{1}{\pi} \arctan x \bigg|_{t_1}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan t_1\right),$$

and

$$0.025 = \int_{t_2}^{\infty} f_2(x)\, dx = \frac{x}{2\sqrt{x^2+2}}\bigg|_{t_2}^{\infty} = \frac{1}{2}\left(1 - \frac{t_2}{\sqrt{t_2^2+2}}\right).$$

Solving these equations gives us

$$\boxed{t_1 = \tan\left(\tfrac{19\pi}{40}\right) \text{ and } t_2 = \sqrt{\tfrac{722}{39}}}.$$

Week 6. *Proposed by Matthew McMullen.*

In order to save time on a group stats project, you and your partner collect data separately. You ask 30 randomly-chosen students what their GPAs are, and you obtain a sample mean of 2.85 and a sample standard deviation of 0.45. Your friend asks 20 randomly-chosen students what their GPAs are, and she obtains a sample mean of 3.05 and a sample standard deviation of 0.55. Based only on this information, find the sample standard deviation of the GPAs of all 50 students.

Solution. Let $\{x_i\}_{i=1}^{30}$ be your data set, and let $\{y_i\}_{i=1}^{20}$ be your friend's data set. Solving the sample standard deviation formula for the sum of squares yields

$$\sum_{i=1}^{30} x_i^2 = 29(0.45)^2 + 30(2.85)^2 = 249.5475$$

and

$$\sum_{i=1}^{20} y_i^2 = 19(0.55)^2 + 20(3.05)^2 = 191.7975.$$

Next, notice that the overall sample mean is $\frac{30(2.85)+20(3.05)}{50} = 2.93$, and let s denote the overall sample standard deviation. Then

$$\begin{aligned} 49s^2 &= \sum_{i=1}^{30}(x_i - 2.93)^2 + \sum_{i=1}^{20}(y_i - 2.93)^2 \\ &= \sum_{i=1}^{30} x_i^2 + \sum_{i=1}^{20} y_i^2 - 2(2.93)(30(2.85) + 20(3.05)) + 50(2.93)^2 \\ &= 12.1. \end{aligned}$$

Thus, $s = \sqrt{\frac{121}{490}} \approx \boxed{0.497}$.

Week 7. *Proposed by Matthew McMullen.*

Suppose $f(x) = \frac{ax}{bx^2+c}$ has an absolute maximum value of 2013 at $x = 2$. Find the coordinates of all of the inflection points of f.

Solution. Since f has an absolute maximum, we may assume, without loss of generality, that $b = 1$ and $c > 0$. Then

$$f'(x) = \frac{a(c - x^2)}{(x^2 + c)^2}$$

has two critical points: $\pm\sqrt{c}$. Since $\lim_{x\to-\infty} f(x) = 0$, the first derivative test, with $a > 0$, tells us that f has an absolute maximum value of $\frac{a}{2\sqrt{c}}$ at $x = \sqrt{c}$. Thus, $c = 4$ and $a = 8052$.

Now we can calculate

$$f''(x) = \frac{16,104x(x^2 - 12)}{(x^2 + 4)^3}$$

and use a sign chart to show that f has three inflection points: $\boxed{(0,0),\ (2\sqrt{3}, 2013\sqrt{3}/2),\text{ and } (-2\sqrt{3}, -2013\sqrt{3}/2)}$.

Week 8. *Proposed by Matthew McMullen.*

The line tangent to the curve $y^3 + 3y = x^2$ at the point $(2, 1)$ intersects the curve at another point. Find the coordinates of this point.

Solution. Using implicit differentiation, we find that

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 3}.$$

Plugging in $x = 2$ and $y = 1$ gives us the slope of the tangent line: $2/3$. Thus, $y = (2/3)x - 1/3$ is the equation of the tangent line. Plugging this into the curve and simplifying, we see that

$$8x^3 - 39x^2 + 60x - 28 = 0,$$

or $(x - 2)^2(8x - 7) = 0$. Thus, $x = 7/8$ and $y = (2/3)(7/8) - 1/3 = 1/4$. The coordinates of the intersection point is $\boxed{(7/8, 1/4)}$.

Week 9. *Proposed by Matthew McMullen.*

Find all pairs of rational numbers (x, y) such that $y^2 = x^3 - 432$. (*Hint:* If $u = \frac{36+y}{6x}$ and $v = \frac{36-y}{6x}$, what is $u^3 + v^3$?)

Solution. Suppose (x, y) is a pair of rational numbers that satisfies $y^2 = x^3 - 432$. Clearly, $x \neq 0$, and we define u and v as in the hint. Then (u, v) is a pair of rational numbers that satisfies $u^3 + v^3 = 1$. By Fermat's Last Theorem for exponent 3, we must have either $u = 0, v = 1$ or $u = 1, v = 0$. Therefore, $(x, y) = \boxed{(12, -36)}$ or $(x, y) = \boxed{(12, 36)}$. Since both of these pairs satisfy the original equation, these are the two solutions.

Week 10. *Proposed by Dave Stucki.*

Show that

$$\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} = 2$$

and

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 4.$$

Solution. Notice that

$$(\sqrt{3}+1)^3 = (\sqrt{3})^3 + 3(\sqrt{3})^2 \cdot 1 + 3\sqrt{3} \cdot 1^2 + 1^3 = 6\sqrt{3} + 10 = \sqrt{108} + 10.$$

Similarly, $(\sqrt{3}-1)^3 = \sqrt{108} - 10$, $(2+i)^3 = 2 + 11i$, and $(2-i)^3 = 2 - 11i$. Thus,

$$\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} = (\sqrt{3}+1) - (\sqrt{3}-1) = 2$$

and

$$\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} = (2+i) + (2-i) = 4.$$

Week 11. *Proposed by Matthew McMullen.*

A drinking glass has a volume of 30 cubic inches. The top edge of the glass is a circle of radius 1.5 in., and the bottom of the glass is an ellipse with major radius 1.25 in. and minor radius 1.0 in. Assuming the top circle linearly tapers to the bottom ellipse, find the height of the glass.

Solution. We first find a general formula for the volume, V, of such a glass with top radius r, height h, bottom major radius a, and bottom minor radius b. Imagine taking cross sections parallel to the top edge. Because the top circle linearly tapers to the bottom ellipse, these cross sections are ellipses with major radius $(a-r)x/h + r$ and minor radius $(b-r)x/h + r$, where x is the distance from the top edge.

Then, with some work, we get

$$\begin{aligned} V &= \int_0^h \pi \left(\frac{a-r}{h}x + r\right)\left(\frac{b-r}{h}x + r\right) dx \\ &= \frac{\pi h}{3}\left(ab + \frac{r}{2}(a+b) + r^2\right). \end{aligned}$$

Plugging in the given information and solving for h gives

$$h = \frac{90}{5.1875\pi} \approx \boxed{5.52 \text{ in.}}$$

Week 12. *Purdue University Problem of the Week.*

A standard six-sided die is rolled forever. Let T_k be the total of all the dots rolled in the first k rolls. Find the probability that one of the T_k's is eight.

Solution. Let $p(k)$ be the probability that $T_k = 8$. Notice that $p(k) = \frac{n(k)}{6^k}$, where $n(k)$ is the number of ways to write 8 as a sum of k integers between 1 and 6, inclusive, where order matters. For example, $n(2) = 5$ since

$$8 = 6 + 2 = 2 + 6 = 5 + 3 = 3 + 5 = 4 + 4.$$

Proceeding in this way, we can find our answer:

$$\frac{5}{6^2} + \frac{21}{6^3} + \frac{35}{6^4} + \frac{35}{6^5} + \frac{21}{6^6} + \frac{7}{6^7} + \frac{1}{6^8} = \frac{450,295}{6^8} \approx \boxed{0.2681}.$$

Week 13. *Purdue University Problem of the Week.*

Let R be the region below the graph of $y = x$ and above the graph of $y = 3^x - x - 1$, for $0 \leq x \leq 1$. Find the volume of the solid obtained by rotating R around the line $y = x$.

Solution. Imagine rotating the solid 45° clockwise, taking slices perpendicular to the x-axis, then rotating the solid back to its original position. If the slicing takes place at $x = \sqrt{2}a$, for $0 \leq a \leq 1$, then the radius of the slice is the distance from the point (a, a) to the point where the lines $y = 2a - x$ and $y = 3^x - x - 1$ meet. Let r_a denote this distance. It can be shown that $r_a^2 = 2(\log_3(2a+1)-a)^2$.

Thus,

$$\begin{aligned} V &= \int_0^1 \pi r_a^2 \, d(\sqrt{2}a) \\ &= 2\sqrt{2}\,\pi \int_0^1 (\log_3(2x+1) - x)^2 \, dx \\ &= 2\sqrt{2}\,\pi \left(\frac{13}{12} - \frac{3}{\ln 3} + \frac{2}{(\ln 3)^2} \right) \\ &\approx \boxed{0.086072}. \end{aligned}$$

Week 14. *Proposed by Matthew McMullen.*

You randomly throw a dart at an equilateral triangle-shaped dartboard. What is the probability that your dart is closer to the center of the dartboard than to any edge?

Solution. Without loss of generality, suppose the height of the triangle is 3, and position it in the plane so that its vertices are the points $(0, -2)$, $(\sqrt{3}, 1)$, and $(-\sqrt{3}, 1)$. Then the origin is the center of the dartboard, and the curve $y = |x|/\sqrt{3}$, together with the negative y-axis, divide it into equal thirds.

If a dart lands on the point (x, y) in the upper third, then it is closer to the center than to any edge if and only if $\sqrt{x^2 + y^2} < 1 - y$, or $y < \frac{1-x^2}{2}$. Notice also that $x = \frac{1}{\sqrt{3}}$ is the positive solution to the equation $\frac{1-x^2}{2} = \frac{x}{\sqrt{3}}$. Thus, due to symmetry, the area of the dartboard that is closer to the center than to any edge is

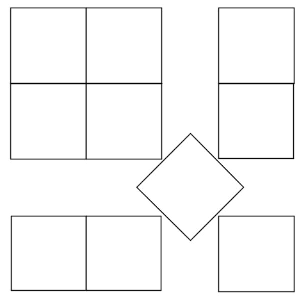
$$6\int_0^{1/\sqrt{3}} \left(\frac{1-x^2}{2} - \frac{x}{\sqrt{3}}\right) dx = \frac{5}{3\sqrt{3}},$$

and the answer to our problem is this area divided by the area of the triangle. Therefore, the probability that the dart lands closer to the center than to any edge is $\frac{5/(3\sqrt{3})}{3\sqrt{3}} = \boxed{\frac{5}{27}}$.

Week 15. *Proposed by Matthew McMullen.*

A rectangular box has dimensions $11.25''$ by $11.25''$ by $3''$. Is it possible to fit ten cubic blocks, each with side length $3''$, in the box? If so, explain how to do it; if not, explain why not.

Solution. Yes! Check out this exactly-to-scale picture:



There are a few other ways to do this problem, too.