Coffee Hour Problems

Edited by Matthew McMullen

Spring 2014

Week 1. *Proposed by Matthew McMullen.*

Find the last two digits of $89^{2014} + 1$.

Week 2. *Proposed by Matthew McMullen.*

Four consecutive even integers are removed from the set $\{1, 2, 3, \ldots n\}$. The average of the remaining numbers is 51.5625. Find the value of n and the values of the integers that were removed.

Week 3. *Proposed by Matthew McMullen.*

Find all pairs of non-negative integers x, y such that $y^2 = x^3 - 3x + 2$.

Week 4. *Proposed by Matthew McMullen.*

Find an explicit continuous function, $f(x)$, such that
(i) f is differentiable for all $x \neq 0$,
(ii) $f(0) = 1/2$,
(iii) $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 0$, and
(iv) $\lim_{x\to 0} \frac{f(x)-1/2}{x} = \infty$.

Week 5. *Proposed by Matthew McMullen.*

Use the fact that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ to find

$$\int_0^1 \int_0^1 \ln(1 - xy)\, dy\, dx.$$

Week 6. *Proposed by Matthew McMullen.*

Let $f(x) = 1 + a_1 \cos(2\pi x) + a_2 \cos(4\pi x)$, where a_1 and a_2 are constants such that $f(x) \geq 0$ for all x. Find the largest possible value of $f(0)$.

Week 7. *Proposed by Matthew McMullen.*

The vertices of a polygon are $(-1, 0)$, $(1, 0)$, (x_1, y_1), and (x_2, y_2), where the last two points are on the top half of the unit circle. Where should these points be placed to maximize the perimeter of the polygon?

Week 8. *Proposed by Matthew McMullen.*

You are standing at the center of the $[-1, 1] \times [-1, 1]$ square.

(a) If you walk off in a random direction, what is the average, or expected, distance you will walk until you hit the edge of the square?

(b) If a point is chosen randomly on the square's edge, what is the average distance from you to the chosen point?

Week 9. *Purdue University Problem of the Week.*

Does the series

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}$$

converge?

Week 10. *Proposed by Matthew McMullen.*

A delivery company will only accept a box for shipment if the sum of its length and girth (distance around) does not exceed some maximum length, M. You would like to ship a box with regular n-gon ends that has the largest possible volume. Show that the length of this box is $M/3$ (regardless of n).

Week 11. *Purdue University Problem of the Week.*

Let f be a positive and continuous function on the real line which satisfies $f(x+1) = f(x)$ for all numbers x. Prove

$$\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})} dx \geq 1.$$

Week 12. *Proposed by Matthew McMullen.*

A one-meter length of wire is cut into three pieces. The first piece is formed into an equilateral triangle, the second piece is formed into a square, and the third piece is formed into a circle. How should the wire be cut to minimize the total area enclosed by these three pieces?

Week 13. *From the 2014 AIME I.*

The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

Week 14. *From the 2014 AIME II.*

The repeating decimals $0.abab\overline{ab}$ and $0.abcabc\overline{abc}$ satisfy

$$0.abab\overline{ab} + 0.abcabc\overline{abc} = \frac{33}{37},$$

where a, b, and c are (not necessarily distinct) digits. Find the three-digit number abc.

Week 15. *(Re)Proposed by Matthew McMullen.*

Is it possible for the sum of two rational numbers to equal the product of their reciprocals? (This question was published some time ago with an incorrect solution!)