Coffee Hour Problems of the Week (solutions)

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Week 1. *Proposed by Matthew McMullen.*

Suppose $a_n > 0$ for all positive integers n and $\lim_{n\to\infty} a_n = 0$. Does it necessarily follow that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges? Why or why not?

Solution. If, in addition, the sequence $\{a_n\}$ is decreasing, the result follows, but this is *not* true in general. For example, consider the sequence defined by $a_n = 1/n$, for n odd, and $a_n = 1/n^2$, for n even. Then for n even, $\sum a_n$ converges, but for n odd, $\sum a_n$ diverges. Thus, in this case, $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges.

Week 2. *Proposed by Matthew McMullen.*

Find

$$\lim_{x\to\infty} \left(x - \sqrt[2013]{x^{2013} + x^{2012}} \right).$$

Solution. Let L be the limit in question. We factor out an x, use the substitution $u = 1/x$, and then apply L'Hospital's Rule to get

$$\begin{aligned} L &= \lim_{x\to\infty} x \left[1 - \left(1 + \frac{1}{x} \right)^{1/2013} \right] \\ &\overset{(u=1/x)}{=} \lim_{u\to 0^+} \frac{1 - (1+u)^{1/2013}}{u} \\ &\overset{(L'H)}{=} \lim_{u\to 0^+} \frac{-1}{2013} (1+u)^{-2012/2013} \\ &= \boxed{\tfrac{-1}{2013}}. \end{aligned}$$

Week 3. *Inspired by a Purdue U. Problem of the Week.*

(a) For integers a, b, and c, show that $a^2 + b^2 + c^2$ cannot be one less than a multiple of 8.

(b) Show that the equation $x^2 + y^2 + z^2 = 7w^2$ has no nontrivial solution over the integers.

Solution. The squares mod 8 are 0, 1, and 4. Thus, $a^2 + b^2 + c^2$ cannot be congruent to 7 mod 8. This solves (a). To solve (b), first note that if w is odd, then $7w^2$ is congruent to 7 mod 8, which contradicts part (a). Suppose $w = 2k$ is the smallest positive integer that solves the given equation. Then $x^2+y^2+z^2$ is a multiple of 4; and so x, y, and z are all even, say $x = 2a_1$, $y = 2a_2$, and $z = 2a_3$. Dividing both sides by 4 yields $a_1^2 + a_2^2 + a_3^2 = 7k^2$, contradicting the minimality of w.

Week 4. *Proposed by Matthew McMullen.*

Given that the periodic continued fraction

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{\cdots}}}}}}}$$

converges, find its value.

Solution. Let x denote the given continued fraction. Then, by repeatedly taking reciprocals and subtracting the appropriate number, we see that

$$x = \frac{1}{\frac{1}{\frac{1}{x}-1} - 2} - 3,$$

which yields

$$x + 3 = \frac{1-x}{3x-2},$$

or $3x^2 + 8x - 7 = 0$. Since x is positive, the quadratic equation gives us

$$x = \boxed{\tfrac{-4+\sqrt{37}}{3}}.$$

Week 5. *Proposed by Matthew McMullen.*

Consider two groups of students: Class 1 and Class 2. Let W_1 be the percentage of women in Class 1 who got an A, W_2 the percentage of women in Class 2 who got an A, M_1 the percentage of men in Class 1 who got an A, and M_2 the percentage of men in Class 2 who got an A. Moreover, let W be the overall percentage of women who got an A, M the overall percentage of men who got an A, C_1 the percentage of Class 1 who got an A, and C_2 the percentage of Class 2 who got an A.

(a) Give an example where $M_1 > W_1$ and $M_2 > W_2$, but $W > M$.

(b) Can you find an example where the condition in (a) is met and a similar condition is exhibited between classes (e.g. $M_1 > M_2$ and $W_1 > W_2$, but $C_1 < C_2$)?

Solution. The main question here is whether or not Simpson's Paradox can hold in both directions in a 2×2 table. For (a), suppose Class 1 has 7 students, 2 men and 5 women, and Class 2 has 12 students, 2 men and 10 women. Then, if $M_1 = M_2 = 50\%$, $W_1 = 40\%$, and $W_2 = 60\%$, we would have $W = 8/15 > 50\% = M$.

The answer to part (b) is, "No," however. Suppose the condition in (a) is met and, without loss of generality, $M_1 > M_2$ and $W_1 > W_2$. We will show that $C_1 > C_2$. The key idea is to note that M, W, C_1, and C_2 are all weighted averages; so, for example, M is between M_1 and M_2. Therefore, the only way all our inequalities can be satisfied is with $M_1 > W_1 > W > M > M_2 > W_2$. But this forces $C_1 > C_2$, since C_i is a weighted average of M_i and W_i for $i = 1, 2$.

Week 6. *Proposed by Matthew McMullen and Ryan Berndt.*

Describe a tiling of the plane with countably many squares such that the sum of the cubes of the side-lengths of all the squares is finite.

Solution. (*Based on an idea of student Stephen Sheneman.*) One way to do this is as follows. Start with one square of side-length 1. Surround this square with eight squares of side-length 1, and divide each of these squares into four congruent squares. Next, surround your current tiling with 16 squares of side-length 1, and divide each of these squares into 16 congruent squares. Continue in this way *ad infinitum.* At the nth step, there will be an additional $8n$ squares of side-length 1 divided into 4^n congruent squares, each having sides of length 2^n. Therefore, the sum of the cubes of the side-lengths of all the squares is given by

$$1 + \sum_{n=1}^{\infty} 8n \cdot 4^n \cdot \left(\frac{1}{2^n}\right)^3 = 1 + 8\sum_{n=1}^{\infty} \frac{n}{2^n} = 17 < \infty.$$

Week 7. *Proposed by Matthew McMullen.*

Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt[2013]{n}}} \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{1}{2} - \frac{1}{2^{\sqrt[n]{n}}} \right).$$

Solution. For the first series, notice that

$$\lim_{n\to\infty} \frac{n^2}{2^{\sqrt[2013]{n}}} = \lim_{u\to\infty} \frac{u^{4026}}{2^u} = 0,$$

where we have made the substitution $n = u^{2013}$. In particular, there is a positive integer N such that

$$\frac{1}{2^{\sqrt[2013]{n}}} < \frac{1}{n^2},$$

for all $n \geq N$. Thus, the first series converges by the comparison test.

For the second series, one can show (using L'Hospital's Rule, for example) that

$$\lim_{n\to\infty} n \left(\frac{1}{2} - \frac{1}{2^{\sqrt[n]{n}}} \right) = \infty.$$

In particular, there is a positive integer N such that

$$\frac{1}{2} - \frac{1}{2^{\sqrt[n]{n}}} > \frac{1}{n},$$

for all $n \geq N$. Thus, the second series diverges by the comparison test.

Week 8. *From David Burton's* Elementary Number Theory.

Find the smallest positive value of n for which
(a) The equation $301x + 77y = 2000 + n$ has a solution over the integers.
(b) The equation $5x + 7y = n$ has exactly three positive solutions over the integers.

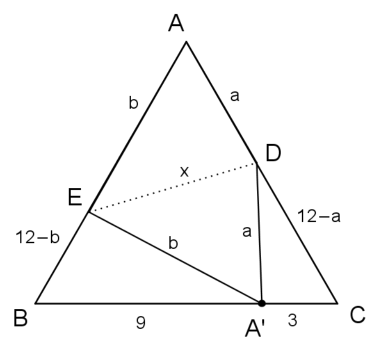
Solution. (a) A necessary and sufficient condition for the existence of a solution to the Diophantine equation $ax + by = c$ is that the greatest common divisor of a and b divides c. Since $\gcd(301, 77) = 7$, we need to find the smallest positive value of n for which 7 divides $2000 + n$. Therefore, $\boxed{n = 2}$. (If you're curious, one solution to the resulting equation is $x = -286, y = 1144$.)

(b) Since $5(3n) + 7(-2n) = n$, all solutions to the given equation are given by $x = 3n - 7t$ and $y = -2n + 5t$, where t ranges through the integers. We see that both x and y are positive if and only if $\frac{2n}{5} < t < \frac{3n}{7}$. Therefore, we need to find the smallest positive value of n such that there are exactly three integers between $2n/5$ and $3n/7$, *exclusive*. Therefore, $\boxed{n = 82}$. (The three solutions in this case are given by $(15, 1)$, $(8, 6)$, and $(1, 11)$.)

Week 9. *Problem 9 on the 2013 AIME I.*

A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. Find the length of the line segment along which the triangle is folded.

Solution. Referring to the diagram below, we can use the Law of Cosines in triangles $A'CD$ and $A'BE$ to get $a = 39/7$ and $b = 39/5$. Then we use the Law of Cosines in triangle EAD to get $x = \boxed{\frac{39\sqrt{39}}{35}}$.



Week 10. *Proposed by Matthew McMullen.*

Let λ be a real number, and, for nonnegative integers k, define

$$p(k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

Find $\sum_{k=0}^{\infty} p(k)$, $\sum_{k=0}^{\infty} k\, p(k)$, and $\sum_{k=0}^{\infty} k^2\, p(k)$.

Solution. First notice that

$$\begin{aligned}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} &= \frac{n!}{k!(n-k)!}\cdot\left(1-\frac{\lambda}{n}\right)^n\cdot\frac{\lambda^k}{(n-\lambda)^k} \\ &= \frac{(1-\lambda/n)^n\cdot\lambda^k}{k!}\cdot\frac{n(n-1)\cdots(n-(k-1))}{(n-\lambda)^k}.\end{aligned}$$

Thus, $p(k) = e^{-\lambda}\lambda^k/k!$.

We then have

$$\sum_{k=0}^{\infty} p(k) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = \boxed{1},$$

$$\sum_{k=0}^{\infty} k\,p(k) = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}e^{\lambda} = \boxed{\lambda}, \text{ and}$$

$$\sum_{k=0}^{\infty} k^2\,p(k) = \lambda\sum_{k=0}^{\infty}(k+1)\frac{e^{-\lambda}\lambda^k}{k!} = \lambda\left(\sum_{k=0}^{\infty} k\,p(k) + \sum_{k=0}^{\infty} p(k)\right) = \boxed{\lambda(\lambda+1)}.$$

(Alternatively, we can recognize $p(k)$ as a Poisson distribution with mean λ (and hence variance λ) and the results follow.)

Week 11. *Proposed by Matthew McMullen.*

Suppose that the functions f and g are differentiable on the real line and that $f'(x) > g'(x) > 0$, for all x. Does it necessarily follow that f is eventually greater than g? In other words, does there exist some M such that
$f(x) > g(x)$, for all $x > M$?

Solution. On first thought, the answer would seem to be "yes." After all, if one car is going faster than another (and they're travelling in the same direction), the faster car will eventually overtake the slower car, regardless of the head start. The problem with this reasoning is that the "overtaking" might take place at infinity.

As a counterexample, consider $f(x) = 2\arctan x$ and $g(x) = \arctan x + \frac{\pi}{2}$. Then $f' > g' > 0$, but $f < g$.

Week 12. *Proposed by Matthew McMullen.*

For positive integers n, define $a_n = \sum_{k=1}^{n} \sin(k)$. Is (a_n) a bounded sequence?

Solution. We will show that the sequence is indeed bounded. Notice that

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \frac{\sin(k)\sin(1/2)}{\sin(1/2)} \\ &= \sum_{k=1}^{n} \frac{\cos(k-1/2)-\cos(k+1/2)}{2\sin(1/2)} \\ &= \frac{\cos(1/2)-\cos(n+1/2)}{2\sin(1/2)}. \end{aligned}$$

Therefore, for all n,

$$\frac{\cos(1/2)-1}{2\sin(1/2)} < a_n < \frac{\cos(1/2)+1}{2\sin(1/2)}.$$

(By Kronecker's Approximation Theorem, these are the best possible bounds.)

Week 13. *Problem 13 on the 2013 AIME II.*

In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, find the area of $\triangle ABC$.

Solution. Let $a = DB$, $b = AE = ED$, and $c = AB$. Moreover, let $x = \cos(\angle AEC)$, $y = \cos(\angle CDE)$, and $z = \cos(\angle DEB)$. Then $\cos(\angle CED) = -x$, $\cos(\angle BDE) = -y$, and $\cos(\angle BEA) = -z$. Using the Law of Cosines in $\triangle AEC$ and $\triangle CED$ gives

$$\begin{aligned} 16a^2 &= 7 + b^2 - 2b\sqrt{7}x \\ 9a^2 &= 7 + b^2 + 2b\sqrt{7}x, \end{aligned}$$

and adding these two equations gives us $25a^2 = 14 + 2b^2$. Using the Law of Cosines in $\triangle CDE$ and $\triangle BDE$ gives

$$\begin{aligned} 7 &= 9a^2 + b^2 - 6aby \\ 9 &= a^2 + b^2 + 2aby, \end{aligned}$$

and adding the first equation to three times the second equation gives us $34 = 12a^2 + 4b^2$. Solving these resulting equations for a and b yields $a = 1$ and $b = \sqrt{11/2}$.

Using the Law of Cosines in $\triangle DEB$ and $\triangle BEA$ gives

$$\begin{aligned} 1 &= \frac{11}{2} + 9 - 6bz \\ c^2 &= \frac{11}{2} + 9 + 6bz, \end{aligned}$$

and adding these equations gives us $1 + c^2 = 11 + 18$, or $c = 2\sqrt{7}$. Therefore, by the Pythagorean Theorem, the altitude from C has length $\sqrt{16 - 7} = 3$, and the area of $\triangle ABC$ is $\boxed{3\sqrt{7}}$.

Week 14. *Proposed by Matthew McMullen.*

A chicken egg with height 6 centimeters is modeled by revolving the curve

$$\frac{x^2}{9} + \frac{y^2}{4} e^{-0.2x} = 1$$

about the x-axis. Find the volume of this egg and its maximum circumference (perpendicular to the x-axis).

Solution. Let V denote the volume of the egg. Solving the given equation for y^2 gives us

$$y^2 = \frac{36 - 4x^2}{9} e^{0.2x}.$$

Using the disk method and integration by parts twice, we have

$$\begin{aligned} V &= \pi \int_{-3}^{3} y^2 \, dx \\ &= \frac{-20\pi}{9} e^{0.2x} (x^2 - 10x + 41) \Big|_{-3}^{3} \\ &= \frac{400\pi}{9} (4e^{-0.6} - e^{0.6}) \\ &\approx \boxed{52.10 \, \mathrm{cm}^3}. \end{aligned}$$

The maximum circumference occurs at the same x-value as the maximum value of y^2. Setting $\frac{d}{dx}(y^2) = 0$ gives us $x_0^2 + 10x_0 - 9 = 0$, or $x_0 = -5 + \sqrt{34}$. Therefore, the maximum circumference is

$$2\pi \frac{\sqrt{36 - 4x_0^2}}{3} e^{0.1x_0} \approx \boxed{13.12 \, \mathrm{cm}}.$$

Week 15. *Proposed by Matthew McMullen.*

A spheroid is generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$ about the x-axis. Suppose the volume of the spheroid is V units cubed and the surface area of the spheroid is S units squared. If $V = S$, find a.

Solution. First notice that $y^2 = \frac{a^2 - x^2}{4}$ and $y' = \frac{-x}{4y}$. Thus,

$$\begin{aligned} V &= 2\pi \int_0^a y^2\, dx \\ &= \frac{\pi}{2} \int_0^a (a^2 - x^2)\, dx \\ &= \frac{\pi a^3}{3}, \end{aligned}$$

and

$$\begin{aligned} S &= 4\pi \int_0^a y\sqrt{1 + (y')^2}\, dx \\ &= 4\pi \int_0^a \sqrt{y^2 + x^2/16}\, dx \\ &= \pi \int_0^a \sqrt{4a^2 - 3x^2}\, dx \\ &= 2a\pi \int_0^a \sqrt{1 - (\sqrt{3}x/2a)^2}\, dx \\ &= \frac{4a^2\pi}{\sqrt{3}} \int_0^{\sqrt{3}/2} \sqrt{1 - u^2}\, du \\ &= \frac{4a^2\pi}{\sqrt{3}} \left(\frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \frac{\pi}{6} \right). \end{aligned}$$

(Where we evaluate the above integral by thinking of it as the area bounded by one-sixth of a unit circle and a right triangle with base $\sqrt{3}/2$ and height $1/2$.) Setting $V = S$ and solving for a, gives $a = \boxed{\frac{3}{2} + \frac{2\pi}{\sqrt{3}}} \approx 5.13$ units.