Coffee Hour Problems of the Week (solutions)

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Week 1. *Proposed by Matthew McMullen.*

A regular hexagon with area 3 is inscribed in a circle. Find the area of a regular hexagon *circumscribed* about the same circle.

Solution. Let r denote the radius of the circle. The inscribed hexagon is comprised of six equilateral triangles with side length r, while the circumscribed hexagon is comprised of six equilateral triangles with *height* r. Thus, the area of the inscribed hexagon is $6 \cdot r/2 \cdot \sqrt{3}r/2$, and the area of the circumscribed hexagon is $6 \cdot \sqrt{3}r/3 \cdot r$. Using the fact that the inscribed hexagon has area 3, we get $12 = 6r^2\sqrt{3}$. Thus, the area of the circumscribed hexagon is $6r^2\sqrt{3}/3 = 12/3 = \boxed{4}$.

Week 2. *Proposed by Matthew McMullen.*

A regular n-gon with area A is inscribed in a circle. Find the area of a regular n-gon *circumscribed* about the same circle (as a function of A and n).

Solution. Let r denote the radius of the circle. Similar to the above solution, we divide each n-gon into n congruent isosceles triangles and use trig to find the side lengths. We find that the area of the inscribed polygon is

$$n \cdot r\sin(\pi/n) \cdot r\cos(\pi/n),$$

while the area of the circumscribed polygon is

$$n \cdot r\tan(\pi/n) \cdot r.$$

Thus, in terms of A and n, the area we seek is given by $\boxed{A\sec^2(\pi/n)}$.

Week 3. *Proposed by Matthew McMullen.*

Can you find two real numbers, a and b, such that $a > 0$ and

$$\int_0^1 (ax+b)\,dx = \int_0^1 (ax+b)^2\,dx = \int_0^1 (ax+b)^3\,dx\,?$$

Solution. Yes! Integrating the above equations gives

$$\frac{a}{2} + b = \frac{a^2}{3} + ab + b^2 = \frac{a^3}{4} + a^2 b + \frac{3ab^2}{2} + b^3.$$

Setting the first part equal to the last part, and factoring, yields

$$(2b+a)(2b^2 + 2ab + a^2 - 2) = 0.$$

If $2b + a = 0$, then we get $a = b = 0$. Thus, we need to solve the system of quadratic equations

$$2b^2 + 2ab + a^2 - 2 = 0 \text{ and } 6b^2 + (6a-6)b + 2a^2 - 3a = 0.$$

We'll leave off the (gory) details and just say that the only solution with $a > 0$ is $\boxed{a = \sqrt{3}}$ and $\boxed{b = (1-\sqrt{3})/2}$.

Week 4. *Proposed by Matthew McMullen.*

Show that

$$\lim_{n\to\infty}\left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln n\right)$$

exists and is between 0.5 and 0.6.

Solution. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$. We will show that the sequence $(H_n - \ln n)$ is decreasing and bounded below by 0.5. Showing this sequence is decreasing is equivalent to showing that $\ln(n+1) - \ln n > \frac{1}{n+1}$, for all n. Put $f(x) = \ln(x+1) - \ln x - 1/(x+1)$. Then $f'(x) = -1/((x(x+1)^2) < 0$, for all $x > 0$. Therefore, for all $x > 0$, $f(x) > \lim_{u\to\infty} f(u) = 0$. In particular, we have shown that $\ln(n+1) - \ln n > \frac{1}{n+1}$, for all positive integers n.

To show that our sequence is bounded below by 0.5, we use the Trapezoid Rule to estimate $\int_1^n (1/x)\,dx$. Since the function $y = 1/x$ is convex on $(0,\infty)$, this will be an overestimate of the integral. Thus,

$$\frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{2}\left(\frac{1}{n-1} + \frac{1}{n}\right) > \int_1^n \frac{1}{x}\,dx = \ln n.$$

In other words, for all $n > 1$,

$$H_n - \ln n > \frac{1}{2} + \frac{1}{2n} > \frac{1}{2}.$$

Since $(H_n - \ln n)$ is decreasing and bounded below by 0.5, we have shown that the limit in question exists and is at least 0.5. To show that the limit does not exceed 0.6, simply note that $H_{22} - \ln 22 = 0.59977\ldots < 0.6$.

Week 5. *Proposed by Ryan Berndt and Matthew McMullen.*

Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Does

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot s_n}$$

converge? What about

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot s_n^2}?$$

Solution. Using last week's problem and the limit comparison test, we see that the first sum is equiconvergent to the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n},$$

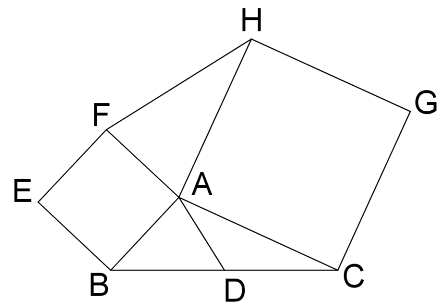
which diverges by the integral test. Similarly, the second sum is equiconvergent to the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2},$$

which converges by the integral test.

Week 6. *Proposed by Zengxiang Tong.*

In the following diagram, $\overline{BD} = \overline{CD}$ and quadrilaterals $ABEF$ and $ACGH$ are squares. Prove $\overline{AD} = \frac{1}{2}\overline{FH}$.



Solution 1. Locate the figure in the plane; where, without loss of generality, $B = (0, 0)$, $D = (1, 0)$, and $A = (x, y)$, for some x, y with $y > 0$. Then it can be shown that $F = (x - y, x + y)$ and $H = (x + y, y - x + 2)$. Then

$$\overline{FH}^2 = (2y)^2 + (2x - 2)^2 = 4(y^2 + (x - 1)^2) = 4\,\overline{AD}^2.$$

Thus, $\overline{AD} = \frac{1}{2}\overline{FH}$.

Solution 2. (*By Stephen Sheneman, Computer Science major.*) Add the point X to the figure so that $ACXB$ is a parallelogram. We are given that $\overline{AC} = \overline{AH}$ and $\overline{CX} = \overline{AB} = \overline{AF}$. Notice that $\angle ACX + \angle CAB = 180^\circ = \angle FAH + \angle CAB$, so $\angle ACX = \angle FAH$. Thus, by SAS, triangles ACX and HAF are congruent; in particular, $\overline{AX} = \overline{FH}$. Since the diagonals of a parallelogram bisect each other, D is on AX and $\overline{AD} = \frac{1}{2}\overline{AX} = \frac{1}{2}\overline{FH}$.

Week 7. *Proposed by Matthew McMullen.*

For any prime $p > 3$, prove that 13 divides $10^{2p} - 10^p + 1$. Can you classify all nonnegative integers n such that 13 divides $10^{2n} - 10^n + 1$?

Solution. In our solution, all of the congruences will be modulo 13. First, since $1001 = 13 \cdot 77$, $10^3 \equiv -1$. Next, since $x^3 + 1 = (x+1)(x^2 - x + 1)$, we have that

$$10^{3n} + 1 = (10^n + 1)(10^{2n} - 10^n + 1), \qquad (1)$$

for all nonnegative integers n. If n is even, (1) implies

$$2 \equiv (10^n + 1)(10^{2n} - 10^n + 1);$$

thus, since 13 is prime, we cannot have $10^{2n} - 10^n + 1 \equiv 0$.

Now suppose $n = 6k + 1$, for some nonnegative integer k. Then (1) implies

$$0 \equiv 11(10^{2n} - 10^n + 1),$$

and it follows that 13 divides $10^{2n} - 10^n + 1$. Similarly, if $n = 6k - 1$, for some positive integer k, (1) implies

$$0 \equiv 5(10^{2n} - 10^n + 1),$$

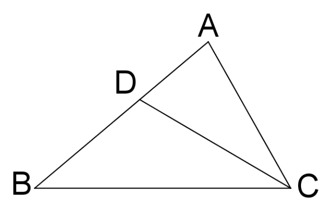
and again 13 divides $10^{2n} - 10^n + 1$. Finally, suppose $n = 6k + 3$, for some nonnegative integer k. Then

$$10^{2n} - 10^n + 1 = (10^6)^{2k+1} - (10^6)^k \cdot 10^3 + 1 \equiv 3.$$

In summary, 13 divides $10^{2n} - 10^n + 1$ if and only if n is either one more or one less than a multiple of 6 (notice that this includes all primes greater than 3).

Week 8. *Proposed by Zengxiang Tong.*

In the following diagram, $\angle BAC = 2\angle ABC$ and CD bisects $\angle ACB$. Show that $\overline{BC} = \overline{CA} + \overline{AD}$.



Solution 1. Extend CA past A to the point X such that $\overline{AX} = \overline{AD}$. Notice that $\angle DAX = \pi - \angle BAC = \pi - 2\angle ABC$. Since $\triangle AXD$ is isosceles, $\angle AXD = (\pi - \angle DAX)/2 = \angle ABC$. Also, we are given that $\angle ACD = \angle BCD$. Thus, by AAS, $\triangle BCD \cong \triangle XCD$. In particular, $\overline{BC} = \overline{XC} = \overline{CA} + \overline{AX} = \overline{CA} + \overline{AD}$.

Solution 2. Let α denote $\angle ABC$, and let β denote $\angle ACD = \angle BCD$. Without loss of generality, suppose $\overline{CA} = 1$. By the Law of Sines applied to $\triangle ABC$,

$$\frac{1}{\sin\alpha} = \frac{\overline{BC}}{\sin 2\alpha} = \frac{\overline{BC}}{2\sin\alpha\cos\alpha}.$$

Thus, $\overline{BC} = 2\cos\alpha$; and we need to show that $\overline{AD} = 2\cos\alpha - 1$.

By the Law of Sines applied to $\triangle CAD$,

$$\frac{\overline{AD}}{\sin\beta} = \frac{1}{\sin(\pi - (2\alpha + \beta))}. \tag{2}$$

We next note that $3\alpha + 2\beta = \pi$, or $\beta = (\pi - 3\alpha)/2$. Thus, (2) yields

$$\begin{aligned}
\overline{AD} &= \frac{\sin(\pi/2 - 3\alpha/2)}{\sin(\pi/2 - \alpha/2)} \\
&= \frac{\cos(3\alpha/2)}{\cos(\alpha/2)} \\
&= \frac{\cos(\alpha + \alpha/2)}{\cos(\alpha/2)} \\
&= \cos\alpha - \sin\alpha\frac{\sin(\alpha/2)}{\cos(\alpha/2)} \\
&= \cos\alpha - 2\sin^2(\alpha/2) \\
&= \cos\alpha + (1 - 2\sin^2(\alpha/2)) - 1 \\
&= 2\cos\alpha - 1.
\end{aligned}$$

Week 9. *Proposed by Matthew McMullen.*

State and prove the general result illustrated by the fact that $4^2 = 16$, $34^2 = 1156$, $334^2 = 111556$, and $3334^2 = 11115556$. Can you find similar results in bases other than 10?

Solution. Let $n \geq 0$. The general result can be represented by

$$(\underbrace{3\ldots3}_{n}4)^2 = \underbrace{1\ldots1}_{n+1}\underbrace{5\ldots5}_{n}6.$$

In summation notation, this can be written as

$$\left(1 + \sum_{k=0}^{n} 3 \cdot 10^k\right)^2 = 1 + \sum_{k=0}^{n} 5 \cdot 10^k + \sum_{k=0}^{n} 10^{k+n+1}.$$

Put $u = \sum_{k=0}^{n} 10^k$. Then we need to show that $(1 + 3u)^2 = 1 + 5u + 10^{n+1}u$. This equation has two solutions: $u = 0$ and $u = (10^{n+1} - 1)/9$, so we're done.

To generalize to base b, we work backwards and let $u = \sum_{k=0}^{n} b^k$. Then $u = (b^{n+1} - 1)/(b - 1)$; or,

$$1 + (b-1)u^2 = b^{n+1}u - u + 1. \tag{3}$$

To complete the square on the left-hand side, we must have $b = a^2 + 1$, for some a. Then (3) is equivalent to

$$(1 + au)^2 = 1 + (2a - 1)u + b^{n+1}u.$$

In other words, for any $a > 1$ and any $n \geq 0$,

$$(\underbrace{a \dots a}_{n}(a+1))^2 = \underbrace{1 \dots 1}_{n+1}\underbrace{(2a-1)\dots(2a-1)}_{n}(2a),$$

in base $a^2 + 1$.

Week 10. *Proposed by Matthew McMullen.*

Show that $\arccos \frac{1}{5} = 2 \arctan \sqrt{\frac{2}{3}}$.

Solution. Put $u = \arctan \sqrt{\frac{2}{3}}$ (so $0 < u < \pi/2$). Then $\tan u = \sqrt{\frac{2}{3}}$, and solving the associated right triangle tells us that $\cos u = \sqrt{\frac{3}{5}}$. Thus,

$$\cos(2u) = 2\cos^2 u - 1 = 2 \cdot \frac{3}{5} - 1 = \frac{1}{5}.$$

Since $0 < 2u < \pi$, $\arccos(\cos(2u)) = 2u$; in other words, $\arccos \frac{1}{5} = 2u = 2 \arctan \sqrt{\frac{2}{3}}$.

Week 11. *Proposed by Matthew McMullen.*

(a) What is the expected number of times you must roll a fair die to get two consecutive sixes?

(b) Your friend bets you that it will take at least 30 rolls for you to get two consecutive sixes. Should you take this bet?

Solution. (a) Let $p(n)$ be the probability that it takes n rolls to get two consecutive sixes. Then $p(1) = 0$ and $p(2) = 1/36$. Suppose $n > 2$. If our first roll is not a six, then we have $n - 1$ rolls to get two consecutive sixes. If our first roll is a six, then our next roll cannot be a six (since $n > 2$), and we have $n - 2$ rolls to get two consecutive sixes. Thus, for $n > 2$,

$$p(n) = \frac{5}{6} \cdot p(n-1) + \frac{5}{36} \cdot p(n-2). \tag{4}$$

We need to find the expected value of n. First, since $p(n)$ is a probability distribution, $\sum_{n=2}^{\infty} p(n) = 1$. Let $x = \sum_{n=2}^{\infty} n \cdot p(n)$. Then, using (4), we have

$$\begin{aligned} x &= 2 \cdot p(2) + \frac{5}{6} \sum_{n=3}^{\infty} n \cdot p(n-1) + \frac{5}{36} \sum_{n=3}^{\infty} n \cdot p(n-2) \\ &= \frac{2}{36} + \frac{5}{6} \sum_{n=2}^{\infty} (n+1) \cdot p(n) + \frac{5}{36} \sum_{n=1}^{\infty} (n+2) \cdot p(n) \\ &= \frac{2}{36} + \frac{5}{6}(x+1) + \frac{5}{36}(x+2). \end{aligned}$$

Solving for x gives an expected value of $\boxed{42}$ rolls.[1]

For (b), we use Mathematica to compute $\Pr(n \geq 30) = 1 - \Pr(n \leq 29) = 1 - \sum_{n=2}^{29} p(n) = 0.500387$. So we are slightly more likely than not to require 30 or more rolls. You should not take your friend's bet!

Week 12. *Proposed by Matthew McMullen.*

Let $x \geq 0$. Show that

$$(1+x)(1+x^2)\cdots(1+x^{23}) \geq (1+x^{12})^{23}.$$

Solution. By canceling $1 + x^{12}$ from both sides and rearranging factors, what we are trying to show is equivalent to

$$(1+x)(1+x^{23}) \cdot (1+x^2)(1+x^{22}) \cdots (1+x^{11})(1+x^{13}) \geq (1+x^{12})^{22}.$$

To prove this, we will show that

$$(1+x^n)(1+x^{24-n}) \geq (1+x^{12})^2, \tag{5}$$

for all $n = 1, 2, \ldots, 11$. After multiplying out, rearranging, and factoring, (5) is equivalent to the statement

$$x^n(x^{12-n} - 1)^2 \geq 0,$$

which is clearly true.[2]

[1] Similarly, one can show that the standard deviation of this probability distribution is $5\sqrt{66} \approx 40.62$.

[2] Moreover, we have shown that equality is attained if and only if $x = 0$ or $x = 1$.

Week 13. *From* The College Mathematics Journal.

Show that
$$\sin^{-1}\left(\frac{x+3}{\sqrt{12+4x^2}}\right)-\sin^{-1}\left(\frac{x-3}{\sqrt{12+4x^2}}\right)$$
is constant for $-1\le x\le 1$.

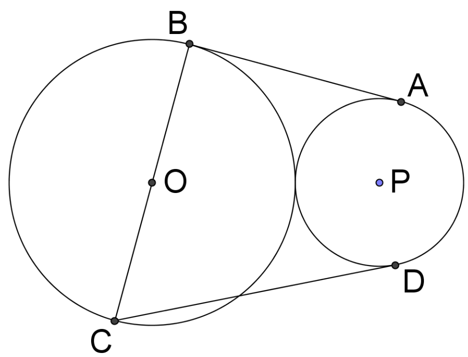
Solution 1. Put
$$f(x)=\sin^{-1}\left(\frac{x+3}{\sqrt{12+4x^2}}\right)-\sin^{-1}\left(\frac{x-3}{\sqrt{12+4x^2}}\right).$$
Then one can show, after a bit of work, that
$$f'(x)=\frac{\sqrt{3}}{x^2+1}\left(\frac{1-x}{|1-x|}-\frac{1+x}{|1+x|}\right).$$
Thus, $f'(x)=0$ for $-1<x<1$. Since f is continuous, this means that f is constant on $[-1,1]$.

Solution 2. Put
$$u=\sin^{-1}\left(\frac{x+3}{\sqrt{12+4x^2}}\right)\quad\text{and}\quad v=\sin^{-1}\left(\frac{x-3}{\sqrt{12+4x^2}}\right).$$
Then $\sin u=\frac{x+3}{\sqrt{12+4x^2}}$, $\sin v=\frac{x-3}{\sqrt{12+4x^2}}$, $\cos u=\frac{\sqrt{3}|1-x|}{\sqrt{12+4x^2}}$, and $\cos v=\frac{\sqrt{3}|1+x|}{\sqrt{12+4x^2}}$. Thus, for $-1\le x\le 1$, we obtain, after a bit of work,
$$\sin(u-v)=\sin u\cos v-\cos u\sin v=\frac{\sqrt{3}}{2}.$$
Since $u-v$ is continuous, we either have $u-v\equiv\pi/3$ or $u-v\equiv 2\pi/3$ on $[-1,1]$.[3]

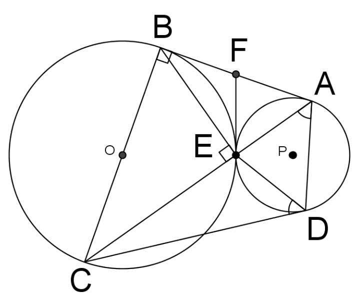
[3]Plugging in $x=0$ shows us that $u-v\equiv 2\pi/3$ on $[-1,1]$.

Week 14. *Proposed by Zengxiang Tong.*

In the following diagram, circles O and P are tangent, AB is tangent to both circles, BC is a diameter of circle O, and CD is tangent to circle P. Show that $\overline{BC} = \overline{CD}$.



Solution. In the diagram below, E is the point of tangency of circles O and P, and EF is tangent to both circles.



Since BC is a diameter, $\angle BEC = 90°$. Since AB is tangent to circle O, $\angle ABC = 90°$. Since FA and FE are tangent to the same circle, $\overline{FA} = \overline{FE}$; hence, $\angle FAE = \angle FEA$. Similarly, $\overline{FB} = \overline{FE}$, and so $\angle FBE = \angle FEB$. We have $180° = 2\angle FEB + 2\angle FEA$. Thus, $\angle AEB = 90°$ and points C, E, and A are collinear.

Now, since triangle ABC is similar to triangle BEC and triangle ADC is similar to triangle DEC, $\overline{BC}^2 = \overline{CA} \cdot \overline{CE} = \overline{CD}^2$, and we are done!

Week 15. *Purdue University Problem of the Week.*

What is the maximum value of a and the minimum value of b for which

$$\left(1+\frac{1}{n}\right)^{n+a} \leq e \leq \left(1+\frac{1}{n}\right)^{n+b}$$

for every positive integer n?

Solution. The given inequality is equivalent to

$$a \leq \frac{1}{\ln(1+1/n)} - n \leq b.$$

Put $f(x) = 1/\ln(1+1/x) - x$. We have two claims: $\lim_{x\to\infty} f(x) = 1/2$ and f is increasing on $(0, \infty)$. Since f is continuous for $x > 0$, proving these claims will show that the maximum value of a is $f(1) = \boxed{1/\ln 2 - 1}$ and the minimum value of b is $\boxed{1/2}$.

For the first claim we have (using little-o notation)

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{1 - x\ln(1+1/x)}{\ln(1+1/x)} \\
&= \lim_{x\to\infty} \frac{1 - x(1/x - 1/(2x^2) + o(1/x^2))}{1/x + o(1/x)} \\
&= \lim_{x\to\infty} \frac{1/(2x) + o(1/x)}{1/x + o(1/x)} \\
&= \frac{1}{2}.
\end{aligned}$$

For the second claim, we will show that $f' > 0$ on $(0, \infty)$. After some manipulation (most notably taking square roots and making the substitution $\sqrt{1+1/x} \mapsto x$), this is equivalent to showing that $2\ln x < x - 1/x$ on $(1, \infty)$. Put $g(x) = x - 1/x - 2\ln x$. Then $g'(x) = 1 + 1/x^2 - 2/x = (1 - 1/x)^2 > 0$ on $(1, \infty)$. Thus, for all $x > 1$, $g(x) > g(1) = 0$; and therefore $f' > 0$ on $(0, \infty)$.