Coffee Hour Problems of the Week

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Otterbein University

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Week 1. *Proposed by Matthew McMullen.*

Suppose $a_n > 0$ for all positive integers n and $\lim_{n\to\infty} a_n = 0$. Does it necessarily follow that $\sum_{n=1}^{\infty}(-1)^n\, a_n$ converges? Why or why not?

Week 2. *Proposed by Matthew McMullen.*

Find

$$\lim_{x\to\infty}\left(x - \sqrt[2013]{x^{2013} + x^{2012}}\right).$$

Week 3. *Inspired by a Purdue U. Problem of the Week.*

(a) For integers a, b, and c, show that $a^2 + b^2 + c^2$ cannot be one less than a multiple of 8.

(b) Show that the equation $x^2 + y^2 + z^2 = 7w^2$ has no nontrivial solution over the integers.

Week 4. *Proposed by Matthew McMullen.*

Given that the periodic continued fraction

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{\cdots}}}}}}}$$

converges, find its value.

Week 5. *Proposed by Matthew McMullen.*

Consider two groups of students: Class 1 and Class 2. Let W_1 be the percentage of women in Class 1 who got an A, W_2 the percentage of women in Class 2 who got an A, M_1 the percentage of men in Class 1 who got an A, and M_2 the percentage of men in Class 2 who got an A. Moreover, let W be the overall percentage of women who got an A, M the overall percentage of men who got an A, C_1 the percentage of Class 1 who got an A, and C_2 the percentage of Class 2 who got an A.

(a) Give an example where $M_1 > W_1$ and $M_2 > W_2$, but $M < W$.

(b) Can you find an example where the condition in (a) is met and a similar condition is exhibited between classes (e.g. $M_1 > M_2$ and $W_1 > W_2$, but $C_1 < C_2$)?

Week 6. *Proposed by Matthew McMullen and Ryan Berndt.*

Describe a tiling of the plane with countably many squares such that the sum of the cubes of the side-lengths of all the squares is finite.

Week 7. *Proposed by Matthew McMullen.*

Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt[2013]{n}}} \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{1}{2} - \frac{1}{2^{\sqrt[n]{n}}} \right).$$

Week 8. *From David Burton's* Elementary Number Theory.

Find the smallest positive value of n for which
(a) The equation $301x + 77y = 2000 + n$ has a solution over the integers.
(b) The equation $5x + 7y = n$ has exactly three positive solutions over the integers.

Week 9. *Problem 9 on the 2013 AIME I.*

A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. Find the length of the line segment along which the triangle is folded.

Week 10. *Proposed by Matthew McMullen.*

Let λ be a real number, and, for nonnegative integers k, define

$$p(k) = \lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda}{n} \right)^k \left(1 - \frac{\lambda}{n} \right)^{n-k}.$$

Find $\sum_{k=0}^{\infty} p(k)$, $\sum_{k=0}^{\infty} k\, p(k)$, and $\sum_{k=0}^{\infty} k^2\, p(k)$.

Week 11. *Proposed by Matthew McMullen.*

Suppose that the functions f and g are differentiable on the real line and that $f'(x) > g'(x) > 0$, for all x. Does it necessarily follow that f is eventually greater than g? In other words, does there exist some M such that
$f(x) > g(x)$, for all $x > M$?

Week 12. *Proposed by Matthew McMullen.*

For positive integers n, define $a_n = \sum_{k=1}^{n} \sin(k)$. Is (a_n) a bounded sequence?

Week 13. *Problem 13 on the 2013 AIME II.*

In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, find the area of $\triangle ABC$.

Week 14. *Proposed by Matthew McMullen.*

A chicken egg with height 6 centimeters is modeled by revolving the curve

$$\frac{x^2}{9} + \frac{y^2}{4}e^{-0.2x} = 1$$

about the x-axis. Find the volume of this egg and its maximum circumference (perpendicular to the x-axis).

Week 15. *Proposed by Matthew McMullen.*

A spheroid is generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$ about the x-axis. Suppose the volume of the spheroid is V units cubed and the surface area of the spheroid is S units squared. If $V = S$, find a.