Coffee Hour Problems of the Week

Edited by Matthew McMullen

Otterbein University

Fall 2012

Week 1. *Proposed by Matthew McMullen.*

A regular hexagon with area 3 is inscribed in a circle. Find the area of a regular hexagon *circumscribed* about the same circle.

Week 2. *Proposed by Matthew McMullen.*

A regular n-gon with area A is inscribed in a circle. Find the area of a regular n-gon *circumscribed* about the same circle (as a function of A and n).

Week 3. *Proposed by Matthew McMullen.*

Can you find two real numbers, a and b, such that $a > 0$ and

$$\int_0^1 (ax+b)\,dx = \int_0^1 (ax+b)^2\,dx = \int_0^1 (ax+b)^3\,dx\,?$$

Week 4. *Proposed by Matthew McMullen.*

Show that

$$\lim_{n\to\infty} \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \ln n \right)$$

exists and is between 0.5 and 0.6.

Week 5. *Proposed by Ryan Berndt and Matthew McMullen.*

Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Does

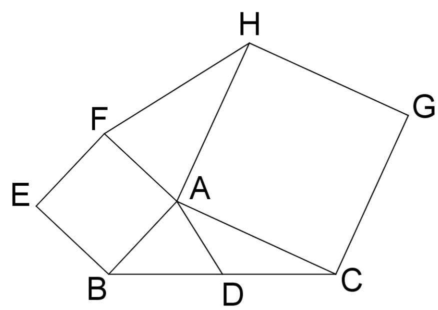
$$\sum_{n=1}^{\infty} \frac{1}{n \cdot s_n}$$

converge? What about

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot s_n^2}?$$

Week 6. *Proposed by Zengxiang Tong.*

In the following diagram, $\overline{BD} = \overline{CD}$ and quadrilaterals $ABEF$ and $ACGH$ are squares. Prove $\overline{AD} = \frac{1}{2}\,\overline{FH}$.

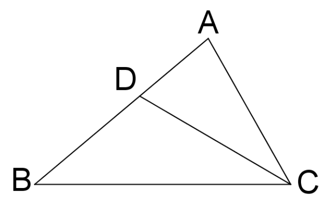


Week 7. *Proposed by Matthew McMullen.*

For any prime $p > 3$, prove that 13 divides $10^{2p} - 10^p + 1$. Can you classify all nonnegative integers n such that 13 divides $10^{2n} - 10^n + 1$?

Week 8. *Proposed by Zengxiang Tong.*

In the following diagram, $\angle BAC = 2\angle ABC$ and CD bisects $\angle ACB$. Show that $\overline{BC} = \overline{CA} + \overline{AD}$.



Week 9. *Proposed by Matthew McMullen.*

State and prove the general result illustrated by the fact that $4^2 = 16$, $34^2 = 1156$, $334^2 = 111556$, and $3334^2 = 11115556$. Can you find similar results in bases other than 10?

Week 10. *Proposed by Matthew McMullen.*

Show that $\arccos \frac{1}{5} = 2 \arctan \sqrt{\frac{2}{3}}$.

Week 11. *Proposed by Matthew McMullen.*

(a) What is the expected number of times you must roll a fair die to get two consecutive sixes?

(b) Your friend bets you that it will take at least 30 rolls for you to get two consecutive sixes. Should you take this bet?

Week 12. *Proposed by Matthew McMullen.*

Let $x \geq 0$. Show that

$$(1+x)(1+x^2)\cdots(1+x^{23}) \geq (1+x^{12})^{23}.$$

Week 13. *From* The College Mathematics Journal.

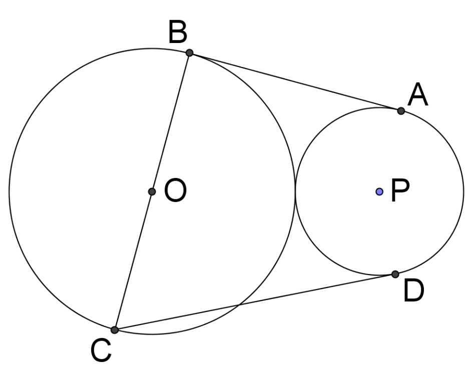
Show that

$$\sin^{-1}\left(\frac{x+3}{\sqrt{12+4x^2}}\right) - \sin^{-1}\left(\frac{x-3}{\sqrt{12+4x^2}}\right)$$

is constant for $-1 \leq x \leq 1$.

Week 14. *Proposed by Zengxiang Tong.*

In the following diagram, circles O and P are tangent, AB is tangent to both circles, BC is a diameter of circle O, and CD is tangent to circle P. Show that $\overline{BC} = \overline{CD}$.



Week 15. *Purdue University Problem of the Week.*

What is the maximum value of a and the minimum value of b for which

$$\left(1+\frac{1}{n}\right)^{n+a} \leq e \leq \left(1+\frac{1}{n}\right)^{n+b}$$

for every positive integer n?