Coffee Hour Problems of the Week (solutions)

Edited by Matthew McMullen

Otterbein University

Spring 2012

Week 1. *Proposed by Matthew McMullen.*

Can you find two rational numbers whose sum and product are reciprocals of each other?

Solution. We will show $\boxed{\text{it is impossible to find two such numbers}}$. Suppose, on the contrary, that $s, t \in \mathbb{Q}$ satisfy $r + s = 1/(rs)$. Multiplying both sides by r and rearranging yields $r^2 + sr - 1/s = 0$. Since r is rational, the discriminant of this quadratic in r must be the square of a rational number. In other words, $s^2 + 4/s = t^2$ for some $t \in \mathbb{Q}$. Multiplying both sides by s and rearranging yields $s^3 - t^2 s + 4 = 0$. By the rational root theorem, $s \in \{1, -1, 2, -2, 4, -4\}$. But this forces $t^2 \in \{5, -3, 6, 2, 17, 15\}$, which yields a contraction since t is rational.

Week 2. *Proposed by Matthew McMullen.*

Find

$$\lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}\, dx.$$

Solution (sketch). Put

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}\, dx.$$

From the substitution $\tan\theta = x$, we get

$$I_n = \int_{-\pi/2}^{\pi/2} \cos^{2n-2}\theta \, d\theta.$$

Using integration by parts, it can be shown that

$$I_{n+1} = \frac{2n-1}{2n} I_n,$$

for $n \geq 1$. Therefore, $I_1 = \pi$ and

$$I_{n+1} = \pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

By the Wallis product formula, we have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n+1} \cdot I_{n+1} &= \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{2n+1}} \cdot \pi \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)} \\
&= \frac{1}{\sqrt{2}} \cdot \pi \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \\
&= \boxed{\sqrt{\pi}}.
\end{aligned}$$

Week 3. *Proposed by Matthew McMullen.*

You have a bag that contains two dice. One is a standard, fair die, and the other is a fair die, but the 1 has been replaced with a 6. You reach into the bag and choose one of the two dice at random, then you roll the die and get a 6. What is the probability that you'll get a six if you roll the same die again?

Solution. Let A be the event that you chose the fair die, B the event that you chose the biased die, and C the event that you roll a 6. By Bayes' Theorem,

$$\begin{aligned}
P(A|C) &= \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|B)P(B)} \\
&= \frac{\frac{1}{6} \cdot \frac{1}{2}}{\frac{1}{6} \cdot \frac{1}{2} + \frac{2}{6} \cdot \frac{1}{2}} \\
&= \frac{1}{3}.
\end{aligned}$$

Thus, $P(B|C) = 2/3$, and the probability you'll get another six is

$$\frac{1}{3} \cdot \frac{1}{6} + \frac{2}{3} \cdot \frac{2}{6} = \boxed{\tfrac{5}{18}}.$$

Week 4. *Proposed by Matthew McMullen.*

Two cubes have rational side-lengths. Their total volume is 19. Find both side-lengths.

Solution. Let C be the curve defined by the equation $x^3 + y^3 = 19$. We need to find a point on C with positive rational coordinates. By inspection, the point $P = (3, -2)$ is seen to be on C. Our idea is to find the equation of the line tangent to C at P, and then to find the other intersection point of this line with C.

By implicit differentiation, $3x^2 + 3y^2y' = 0$; thus, $y' = -x^2/y^2$. The slope of C at P is therefore $-9/4$, and the equation of the line tangent to C at P is $y = -9(x-3)/4 - 2$. After some tedious algebra, we find that this line intersects C at P and at the point $(33/35, 92/35)$. Thus, one possible solution[1] is $\boxed{\frac{33}{35} \text{ and } \frac{92}{35}}$.

Week 5. *Proposed by Matthew McMullen.*

For $a, b > -100$, define $a \oplus b$ to be the effective percent change of an $a\%$ change followed by a $b\%$ change. For example, $10 \oplus -15 = -6.5$, since 100 increased by 10% is 110, and 110 decreased by 15% is 93.5. Is the interval $(-100, \infty)$ a commutative group under $\oplus$?

Solution. $\boxed{\text{Yes}}$. 100 increased by $a\%$ is $100 + a$ and $100 + a$ increased by $b\%$ is

$$\begin{aligned} (100+a) + \frac{b}{100}(100+a) &= (100+a)(1+b/100) \\ &= \frac{(100+a)(100+b)}{100}. \end{aligned}$$

Thus,

$$\begin{aligned} a \oplus b &= \frac{(100+a)(100+b)}{100} - 100 \qquad (1) \\ &= a + b + \frac{ab}{100}. \qquad (2) \end{aligned}$$

Let $I = (-100, \infty)$. Using (1), it is clear that I is closed under $\oplus$. Using (2), we see immediately that $a \oplus b = b \oplus a$ and that $a \oplus 0 = 0$, for all $a, b \in I$. Solving $a + b + ab/100 = 0$, for b, gives us

$$a^{-1} = \frac{-100a}{100+a},$$

and a quick calculation shows that

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) = a + b + c + \frac{ab + ac + bc}{100} + \frac{abc}{10,000}.$$

[1] Are there others?

Week 6. *Purdue University Problem of the Week (paraphrased).*

Let S be a finite set, and let $P(S)$ denote the power set of S. Suppose that $*$ is a function from $P(S)$ to itself. For every $A \in P(S)$, let A^* denote the image of A under $*$. If $A^* \subseteq B^*$ whenever $A \subseteq B \subseteq S$, show that $*$ has a fixed point. Does this conclusion necessarily hold if S is infinite?

Solution. Suppose $*$ has no fixed points. Define $S_0 = S$ and $S_n = S_{n-1}^*$, for all positive integers n. By induction and the fact that $*$ has no fixed points, it is clear that $S_n \subsetneq S_{n-1}$ for all n. This gives us the infinite chain of (strict!) containments:

$$S \supsetneq S_1 \supsetneq S_2 \supsetneq \ldots,$$

which contradicts the fact that S is finite.

The above argument breaks down if S is infinite, but, surprisingly, the conclusion holds regardless of the cardinality of S! Can you prove this?

Week 7. *Seen on a T-shirt.*

Let $\overline{AB}$ and $\overline{CD}$ be two chords of a circle. Suppose these chords are perpendicular and meet at the point P inside the circle. If $AP = 2$, $PD = 3$, and $PB = 6$, find the diameter of the circle.

Solution. Locate the circle in the plane so that $A = (0, 0)$, $B = (8, 0)$, and $D = (2, -3)$. We can use these three points to find the equation of the circle: $(x - 4)^2 + (y - 1/2)^2 = 65/4$. Therefore, the diameter of the circle is $\boxed{\sqrt{65}}$.

Week 9. *Proposed by Duane Buck.*

Find the distance from the point (x, y) to the line segment with endpoints (x_1, y_1) and (x_2, y_2).

Solution (*by Dave Stucki*)**.** The idea is to let (x_3, y_3) be the perpendicular projection of (x, y) onto the line containing the given line segment. If $y_1 = y_2$, then $(x_3, y_3) = (x, y_1)$. If $x_1 = x_2$, then $(x_3, y_3) = (x_1, y)$. Otherwise, let $m = \frac{y_2 - y_1}{x_2 - x_1}$. Then it can be shown that

$$x_3 = \frac{x/m + y + mx_1 - y_1}{m + 1/m} \text{ and } y_3 = m(x_3 - x_1) + y_1.$$

Given two points P and Q, let $d(P, Q)$ be the Euclidean distance between them. If

$$d((x_1, y_1), (x_3, y_3)) + d((x_3, y_3), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2)),$$

then (x_3, y_3) is on the given line segment and the distance we seek is $d((x, y), (x_3, y_3))$. Otherwise, the distance we want is given by

$$\min\{d((x_1, y_1), (x, y)), d((x_2, y_2), (x, y))\}.$$

Week 10. *Proposed by Matthew McMullen.*

My 30-year mortgage has a fixed APR of 5.25% compounded monthly. In what month will my payment toward principal exceed my interest payment for the first time?

Solution. We solve this problem in a more general case and leave off several details. Let P be the original loan amount, n the number of times per year interest accrues, r the APR (compounded n times per year), and Y the length of the loan (in years). It can be shown that the interest due in month t is

$$I(t) := P \cdot \frac{r}{n} \cdot \frac{1 - (1 + r/n)^{t-1-nY}}{1 - (1 + r/n)^{-nY}}.$$

We want to find the first month when this interest is less than half the monthly payment. Solving

$$I(t) = \frac{P(r/n)}{2[1 - (1 + r/n)^{-nY}]}$$

for t and rounding up to the next integer yields

$$t = \left\lceil 1 + nY - \frac{\log 2}{\log(1 + r/n)} \right\rceil.$$

Plugging in $Y = 30$, $n = 12$, and $r = .0525$ gives us the answer: $\boxed{\text{month } 203}$.

Week 11. *Proposed by Matthew McMullen.*

One prepayment strategy for installment loans is to pay one-and-a-half times the monthly payment each month. According to my realtor, this cuts the length of a 30-year mortgage roughly in half.

(a) For what fixed APR, compounded monthly, is this *exactly* true?

(b) At my fixed APR of 5.25%, compounded monthly, how long will it take me to pay off my 30-year mortgage using this strategy?

Solution. Let r be the APR and t be the length of time, in years, it takes to pay off the loan. For this prepayment strategy, we have that

$$\frac{1.5}{1-(1+r/12)^{-360}} = \frac{1}{1-(1+r/12)^{-12t}}.$$

For (b), we plug in $r = .0525$ and solve for t (using logarithms) to get $\boxed{\text{14 years, 5 months}}$.

For (a), we plug in $t = 15$ and put $u = (1+r/12)^{-180}$. Then $u < 1$ and

$$\frac{1.5}{1-u^2} = \frac{1}{1-u},$$

or $(2u-1)(u-1) = 0$. Thus $u = 1/2$, which means $r = 12(2^{1/180}-1) \approx \boxed{4.630\%}$.

Week 12. *Proposed by Matthew McMullen.*

Find $p > 1$ such that

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots} = 2012.$$

Solution. Put $S_p = \sum_{n=0}^{\infty} 1/n^p$. Then

$$\begin{aligned} 1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots &= S_p - 2\sum_{n=1}^{\infty}\frac{1}{(2n)^p} \\ &= S_p - 2^{1-p}\cdot S_p. \end{aligned}$$

Thus, we wish to solve

$$2012 = \frac{S_p}{S_p(1-2^{1-p})}.$$

Using logarithms, we see that

$$p = \frac{\ln 2012 - \ln 2011}{\ln 2} + 1 \approx \boxed{1.000717}.$$

Week 13. *Purdue U. Problem of the Week.*

A tetrahedron has a base which is an equilateral triangle of edge length one. The other three faces are congruent isosceles triangles with one edge an edge of the base and the other edges of length three. Find the length of the shortest path, lying in the union of the three isosceles faces, which starts and ends at the same vertex of the base and which meets every line segment drawn from the top vertex to the perimeter of the base triangle.

Solution. Starting at a base vertex, at some point our path must meet the other two edges of length three. The shortest path, therefore, must travel straight across a face to a point on a longer edge, then straight across the adjacent face to a point on another longer edge, then straight across the next adjacent face to the starting point.

Let x and y be the respective distances (along a longer edge) from where the path meets a longer edge to the nearest base vertex. Then, using Law of Cosines, the total length of the path is given by

$$\begin{aligned} D(x,y) &= \sqrt{x^2+1-2x\cos\theta} + \sqrt{y^2+1-2y\cos\theta} \\ &+ \sqrt{(3-x)^2+(3-y)^2-2(3-x)(3-y)\cos(\pi-2\theta)}, \end{aligned}$$

where θ is the base angle in the 3-3-1 triangle. Since $\cos\theta = 1/6$, our distance equation reduces to

$$\begin{aligned} D(x,y) &= \sqrt{x^2+1-x/3} + \sqrt{y^2+1-y/3} \\ &+ \sqrt{x^2+y^2+1-x/3-y/3-17xy/9}. \end{aligned}$$

The minimum value of this function on the set $[0,3] \times [0,3]$ is $D(1/3, 1/3) = \boxed{26/9}$.

Week 14. *Proposed by Matthew McMullen.*

Let $r, s > 0$. Find the parabola that opens down, goes through the points $(0, 0)$ and (r, s), and cuts off the minimum possible area in the first quadrant. What is this minimum area?

Solution. Suppose the parabola has the form $y = -ax^2 + bx$, for some a, b with $a > 0$. Then its roots are 0 and b/a; and, since the point (r, s) is on the parabola, we have $a = (br - s)/r^2$. Since $a > 0$, one consequence of this last equation is the fact that $b > s/r$.

Let A be the area cut off in the first quadrant. Then

$$A = \int_0^{b/a} (-ax^2 + bx)\,dx,$$

which can be written as a function of b:

$$A(b) = \frac{r^4}{6} \cdot \frac{b^3}{(br - s)^2}.$$

We see that

$$\frac{dA}{db} = \frac{r^4}{6} \cdot \frac{b^2(br - 3s)}{(br - s)^3}.$$

The only critical value of this function on the interval $(s/r, \infty)$ is $b = 3s/r$, which can be shown (using, say, the first derivative test) to be the location of the minimum value of A. The parabola we seek is given by

$$y = \frac{-2s}{r^2}x^2 + \frac{3s}{r}x,$$

and the minimum area is $A(3s/r) = \boxed{\text{9rs/8}}$.

Week 15. *From* The College Mathematics Journal.

Let $ABCD$ be a square with sides of length 1. Suppose K and L are points on the sides BC and CD, respectively, so that the perimeter of triangle KCL is 2. If triangle AKL has minimum area, determine the measures of its angles.

Solution. Let x be the length of LC, and let y be the length of KC. Using the fact that $x + y + \sqrt{x^2 + y^2} = 2$, we see that $y = 2(1 - x)/(2 - x)$. Let $A(x)$ denote the area of triangle AKL. Then it can be shown that

$$A(x) = 1 - \frac{1 - x + xy + 1 - y}{2} = \frac{x^2 - 2x + 2}{2(2 - x)}.$$

We need to minimize this function on the interval $(0, 1)$. We see that

$$A'(x) = \frac{-x^2 + 4x - 2}{2(2 - x)^2};$$

so the critical value of $A(x)$ occurs when $x^2 - 4x + 2 = 0$, or $x = 2 - \sqrt{2}$. Using the first-derivative test, this is seen to give the minimum value of $A(x)$. Now, $x = 2 - \sqrt{2}$ corresponds to $y = 2 - \sqrt{2}$, so triangle AKL is isosceles. Let M be the midpoint of KL. Then triangles AMK and ABK are congruent. Thus, the base angles in triangle AKL are $(\pi - \pi/4)/2 = \boxed{3\pi/8}$ radians, and the remaining angle is $\boxed{\pi/4}$ radians.

Summer! *Classic unsolved problem from 2009.*

A lock consists of exactly n buttons. The lock is opened by pressing all of the buttons in some unknown order. The lock doesn't reset if the wrong order is tried. Assuming optimal strategy, what is the maximum number of button-presses that are needed to open the lock?

Solution. The conjecture is that it will take at most

$$\sum_{k=1}^{n} k!$$

button-presses, but I have not been able to prove this. Let me know if you find a solution!