Coffee Hour Problems of the Week (solutions)

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Week 1. *Proposed by Matthew McMullen.*

(a) Three boxes of fruit are labeled *Apples*, *Oranges*, and *Apples and Oranges*. Each label is wrong. By selecting just one fruit from just one box, how can you determine the correct labeling of the boxes?

(b) There are two boxes: one marked A and one marked B. Each box contains either \$1 million or a deadly snake that will kill you instantly (but not both). You must open one box. On box A there is a sign that reads: "At least one of these boxes contains \$1 million." On box B there is a sign that reads: "A deadly snake that will kill you instantly is in box A." You are told that either both signs are true or both are false. Which box do you open?

Solution. (a) Pick a fruit out of the box labeled *Apples and Oranges*. If it's an apple, you know that this box contains just apples. Then the box labeled *Oranges* cannot contain just apples and cannot contain just oranges (remember: each label is wrong), so it must contain apples and oranges. Finally, the box labeled *Apples* must contain just oranges. A similar argument solves the problem if you draw an orange from the *Apples and Oranges* box.

(b) Suppose both signs are false. Then neither of the boxes contain \$1 million and there is not a snake in box A. So neither of the boxes contain \$1 million and box A contains \$1 million, a contradiction. Therefore, both signs are true, and there is a deadly snake in box A and \$1 million in box B.

Week 2. *Proposed by Matthew McMullen.*

Baseball player Adam Dunn is having a horrible year. According to an article in the August 28, 2011 *Columbus Dispatch*, "The Chicago White Sox slugger possessed a .165 average through Thursday, needing hits in 16 consecutive at-bats just to pass the Mendoza line of .200." How many hits and how many at-bats did Dunn have through Thursday, August 25, 2011?

Solution. Let y be the number of hits and x be the number of at-bats for Dunn through Thursday (so $x > y > 0$ and x and y are integers). We are given three inequalities:

$$0.1645 \le \frac{y}{x} < 0.1655,$$
$$\frac{y+16}{x+16} \ge 0.2005, \text{ and}$$
$$\frac{y+15}{x+15} < 0.2005.$$

Combining the first and second inequalities yields $x \le 365$, and combining the first and third inequalities yields $x \ge 334$. Running through each of the 32 possible values for x, we see that there are actually 8 possible pairs (x, y) that satisfy all three inequalities:

$$(340,56),(345,57),(346,57),(351,58),(352,58),(357,59),(358,59), \text{ and } (363,60).$$

(Consulting the stats shows that Dunn actually had 60 hits in 363 at-bats through Thursday, August 25, 2011.)

Week 3. *Proposed by Matthew McMullen.*

Find
$$\lim_{x\to\infty}\left(e^{-2011}\left(1+\frac{2011}{x}\right)^x\right)^x.$$

Solution. Put
$$f(x)=\left(e^{-2011}\left(1+\frac{2011}{x}\right)^x\right)^x.$$

Then,

$$\begin{aligned}
\ln f(x) &= \frac{\ln\left[e^{-2011}\left(1+\frac{2011}{x}\right)^x\right]}{1/x} \\
&= \frac{-2011 + x\ln\left(1+\frac{2011}{x}\right)}{1/x} \\
&= \frac{-2011 + x\left(\frac{2011}{x} - \frac{2011^2}{2x^2} + o\left(\frac{1}{x^2}\right)\right)}{1/x} \\
&= \frac{-2011^2}{2} + o(1).
\end{aligned}$$

Therefore, the limit we seek is $\boxed{e^{-2011^2/2}}$.

Week 4. *Proposed by Matthew McMullen.*

Is it possible to find a strictly increasing, unbounded sequence of positive numbers, (x_n), such that

$$\sum_{n=1}^{\infty}\left(1-\frac{x_n}{x_{n+1}}\right)$$

converges?

Solution. Suppose that such a sequence exists. Then we must have

$$\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=1.$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\frac{\left(1-\frac{x_n}{x_{n+1}}\right)}{\left(\frac{x_{n+1}}{x_n}-1\right)} &= \lim_{n\to\infty}\frac{x_n}{x_{n+1}}\\ &= 1.\end{aligned}$$

Therefore, by the limit comparison test,

$$\sum_{n=1}^{\infty}\left(\frac{x_{n+1}}{x_n}-1\right)$$

converges. But this sum is a left-hand Riemann sum for $\int_{x_1}^{\infty}1/x\,dx$, and so it must diverge. Therefore, no such sequence (x_n) exists.

Week 5. *Proposed by Matthew McMullen.*

Let $P(x)$ be a polynomial of degree 2011 satisfying $P(k)=k$, for all $k=1,2,\ldots,2011$, and $P(0)=2$. Find $P(-2)$.

Solution. The polynomial $P(x)-x$ has degree 2011 and its roots are all of the integers from 1 to 2011. Thus,

$$P(x)=x+c\prod_{k=1}^{2011}(x-k),$$

for some constant c.

We are given that

$$2 = P(0) = c\prod_{k=1}^{2011}(-k) = -c \cdot 2011!.$$

Therefore, $c = -2/2011!$; and so,

$$\begin{aligned} P(-2) &= -2 - \frac{2}{2011!}\prod_{k=1}^{2011}(-2-k) \\ &= -2 + \frac{2 \cdot 3 \cdot 4 \cdots 2011 \cdot 2012 \cdot 2013}{1 \cdot 2 \cdot \cdots 2011} \\ &= -2 + 2012 \cdot 2013 \\ &= \boxed{4{,}050{,}154}. \end{aligned}$$

Week 6. *Proposed by Jeremy Moore.*

Let $\mathbf{R}$ be the set of all real numbers. Then $M_n(\mathbf{R})$ denotes the set of all $n \times n$ matrices with real-number entries. Consider this set as a vector space over $\mathbf{R}$. Let $n = 2$, and suppose you have a collection of linearly independent, invertible matrices in this vector space. Are the inverses of these matrices linearly independent? What if $n > 2$?

Solution. Let $M_1, \ldots, M_k$ be linearly independent, invertible matrices in $M_2(\mathbf{R})$, and suppose

$$\alpha_1 M_1^{-1} + \cdots + \alpha_k M_k^{-1} = \mathbf{0}, \tag{1}$$

where each $\alpha_i \in \mathbf{R}$ and $\mathbf{0}$ represents the 2×2 zero matrix. Recall that if $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, then $M_i^{-1} = (a_i d_i - b_i c_i)^{-1}\begin{pmatrix} d_i & -b_i \\ -c_i & a_i \end{pmatrix}$. Therefore, (1) implies

$$\sum_{i=1}^{k} \frac{\alpha_i}{a_i d_i - b_i c_i}(*)_i = 0,$$

where $(*)$ can represent a, b, c, d. Since the M_i's are linearly independent, this means that each $\alpha_i/(a_i d_i - b_i c_i) = 0$. Therefore, $\alpha_i = 0$ for all i. Thus, the inverses of the original matrices are linearly independent.

For any $n > 2$, Jeremy tells me that the answer is "no"! You'll have to ask him to supply a counterexample (`jmoore@otterbein.edu`).

Week 7. *Proposed by Devin Fraze.*

You have just solved a Sudoku, but your friend doesn't believe you. Since your friend would like to try the same puzzle later, how can you prove to her that you have solved it without giving away any specific information about the solution?

Solution. One way to do this is with the help of playing cards. In private, lay out your solution using several decks of cards, and then turn face down all of the numbers that aren't given in the original problem. Now invite your friend in the room so she can see that the given numbers are in the right place. Next, pick up all the cards in the first row, shuffle them, and turn them face up so your friend can see that each of the numbers from 1 to 9 are represented. Then replace these cards in their original places (making sure your friend doesn't peek!), and repeat this process for the remaining rows, columns, and 3 by 3 blocks.

We assume, of course, that you really have solved this Sudoku and are not cheating at any stage of this process!

Week 8. *Proposed by Matthew McMullen.*

Imagine a game where six fair dice are rolled. If exactly four different numbers appear, you win; otherwise, you lose. What are your odds of winning this game?

Solution. There are two ways to get exactly four different numbers: three-of-a-kind and three all different, or two pairs and two different. In the first case, there are 6 choices for the repeated number and $\binom{6}{3}$ slots where it can go. For the other slots, we have $5 \cdot 4 \cdot 3$ options. In the second case, we have $\binom{6}{2}$ choices for the two numbers with pairs. The first pair has $\binom{6}{2}$ slots where it can go, and the second pair has $\binom{4}{2}$ slots where it can go. For the other slots, we have $4 \cdot 3$ options.

Therefore, the probability of winning is

$$\frac{6 \cdot \binom{6}{3} \cdot 5 \cdot 4 \cdot 3 + \binom{6}{2}\binom{6}{2}\binom{4}{2} \cdot 4 \cdot 3}{6^6} = \boxed{\frac{325}{648}},$$

which is, surprisingly, greater than 1/2.

Week 9. *Proposed by Matthew McMullen.*

Imagine an experiment where six fair dice are rolled. The random variable x represents the number of different numbers that appear (so $x = 1, \ldots, 6$). Find the probability distribution for x, the expected value of x, and the standard deviation of x.

Solution. Let $p(x)$ be the probability that exactly x different numbers appear. From the previous CHP, $p(4) = 325/648$. Clearly, $p(1) = 1/6^5 = 1/7776$ and $p(6) = 6!/6^6 = 5/324$. The only way to get five different numbers is if you have exactly one pair. We have 6 choices for the pair, $\binom{6}{2}$ slots where it can go, and $5 \cdot 4 \cdot 3 \cdot 2$ options for the other slots. Thus, $p(5) = 15 \cdot 6!/6^6 = 25/108$.

There are five ways to get two different numbers: five-of-a-kind, four-of-a-kind and one pair, and two three-of-a-kinds. In the first case, we have 6 choices for the repeated number, $\binom{6}{5}$ slots where it can go, and 5 options for the remaining slot. In the second case, we have 6 choices for the four-of-a-kind number, $\binom{6}{4}$ slots where it can go, and 5 options for the remaining two slots. In the last case, we have $\binom{6}{2}$ options for the two repeated numbers, and $\binom{6}{3}$ slots where the first number can go. Thus,

$$p(2) = \frac{6 \cdot 5 \cdot \binom{6}{5} + 6 \cdot 5 \cdot \binom{6}{4} + \binom{6}{2}\binom{6}{3}}{6^6} = \frac{155}{7776}.$$

Finally, since the total probability must be 1, we have $p(3) = 25/108$[1].

Therefore,

$$E(x) = \sum_{n=1}^{6} n \cdot p(n) = \frac{31,031}{7776} \approx 3.991$$

and

$$\sigma = \sqrt{\sum_{n=1}^{6} (n - E(x))^2 \cdot p(n)} \approx 0.778.$$

Week 10. *Taken from Stewart's* Calculus.

Find all pairs of points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

Solution. Since $y' = 4x^3 - 4x - 1$, the equation of the line tangent to y at $x = a$ is given by $l : y = (4a^3 - 4a - 1)(x - a) + (a^4 - 2a^2 - a)$. Suppose the tangent line at $x = b \neq a$ is the same as l. Then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-a(4a^3 - 4a - 1) + a^4 - 2a^2 - a = -b(4b^3 - 4b - 1) + b^4 - 2b^2 - b$. The first equation simplifies to $a^2 + ab + b^2 = 1$, while the second equation simplifies to $3(b^2 + a^2)(b + a) = 2(b + a)$.

Suppose $a + b \neq 0$. Then the second equation above yields $a^2 + b^2 = 2/3$; and so the first equation above gives $ab = 1/3$, or $b = 1/(3a)$. But then the first

[1] Is there some intuitive reason why this is equal to $p(5)$?

equation reduces to $(3a^2-1)^2=0$, or $a=\pm 1/\sqrt{3}$; which in turn tells us that $b=\pm 1/\sqrt{3}=a$, a contradiction. Therefore, $a+b=0$ and we see that $a=\pm 1$ and $b=\mp 1$. So the only two points on y that have a common tangent line are $\boxed{(1,-2) \text{ and } (-1,0)}$.

Week 11. *Proposed by Matthew McMullen.*

Suppose that z is a non-zero complex number. Classify all complex numbers w such that

$$\frac{1}{z+w}=\frac{1}{z}+\frac{1}{w}.$$

Solution. The above equation can be rewritten as $z^2+zw+w^2=0$. If $w=z$, then we would get $z=0$, a contradiction. Thus, $w\neq z$, and we can multiply each side of our equation by $z-w$ to get $z^3-w^3=0$. Therefore, w is z times a non-real cube root of unity. Geometrically speaking, the two solutions to the original equation (where z is fixed) are z rotated $\pm 120^\circ$ about the origin.

Week 12. *Proposed by Matthew McMullen.*

Find all real numbers a such that $(\arcsin\sqrt{a}\,)^2$ is real. (By $w=\arcsin z$, we mean the principle value of the complex extension of $y=\arcsin x$.)

Solution. The above expression is obviously real for $0\le a\le 1$. By solving $\sin\theta=(e^{i\theta}-e^{-i\theta})/(2i)=z$ for θ, we have $\arcsin z=-i\ln(iz+\sqrt{1-z^2})$, where the principle values are taken. Thus,

$$(\arcsin\sqrt{a})^2=-\left[\ln(i\sqrt{a}+\sqrt{1-a})\right]^2. \tag{2}$$

If $a<0$, then the right-hand-side of (2) is $-[\ln(\sqrt{1-a}-\sqrt{-a})]^2$, which is real. If $a>1$, then the right-hand-side of (2) is

$$\begin{aligned} -\left[\ln(i\sqrt{a}+i\sqrt{a-1})\right]^2 &= -\left[\ln\left(i(\sqrt{a}+\sqrt{a-1})\right)\right]^2 \\ &= -\left[\ln(\sqrt{a}+\sqrt{a-1})+i\frac{\pi}{2}\right]^2, \end{aligned}$$

which is not real. Thus $\boxed{a\le 1}$. (Moreover, it can be shown that $(\arcsin z)^2$ is real if and only if either $-1\le z\le 1$ or $\Re(z)=0$.)

Week 13. *Ancient Chinese problem.*

A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. In the ensuing brawl over who should get the extra coins, one pirate was killed. The wealth was redistributed, but this time an equal division left 10 coins. Again an argument developed in which another pirate was killed. But now the total fortune was evenly distributed among the survivors. What was the least number of coins that could have been stolen?

Solution. Let n be the number in question. We are given that $n = 17k + 3$, $n = 16l+10$, and $n = 15m$, for some positive integers k, l, m. Combining the last two equations and reducing modulo 16 gives us $m \equiv 6 \pmod{16}$. Thus, $m = 16a + 6$ for some nonnegative integer a, and we conclude that $n = 15(16a + 6)$. Combining this with the first equation above and reducing mod 17 gives us $a \equiv -1 \pmod{17}$. The smallest possible value of a, then, is 16; and thus, $n = 15(16^2 + 6) = \boxed{3930}$. (In general, all possible number of coins is given by the formula $30(136b - 5)$, where b is any positive integer.)

Week 14. *Proposed by Matthew McMullen.*

Determine all y such that

$$2011^{y/x} = \frac{2011^y}{x}$$

for exactly one x.

Solution. Since the left-hand side of the above equation is always positive, $x > 0$. Multiplying through by x and raising both sides to the power of x gives us

$$x^x \cdot 2011^y = 2011^{xy}.$$

Taking the log of both sides and rearranging gives us

$$x \ln x = (y \ln 2011)x - y \ln 2011.$$

Put $u = y \ln 2011$. Then $x \ln x = ux - u$. This equation obviously has $x = 1$ as a solution, and there are no other solutions if and only if either $u \leq 0$ or $u = 1$ (since $y = x - 1$ is tangent to the convex curve $y = x \ln x$ at the point $(1, 0)$). Therefore, either $y \leq 0$ or $y = 1/\ln 2011$.

Week 15. *Proposed by Matthew McMullen.*

Find the number of ordered pairs of integers, (x, y), such that

$$2011 \leq \sqrt{x} + \sqrt{y} \leq 2012.$$

Solution. Obviously, $0 \leq x, y \leq 2012^2$. If $y > 2011^2$, then x must be 0. Therefore, there are $2012^2 - 2011^2 = 4023$ ordered pairs with $y > 2011^2$. If $y \leq 2011^2$, then our inequality is equivalent to

$$(2011 - \sqrt{y})^2 \leq x \leq (2012 - \sqrt{y})^2.$$

This means that x must be one of $\lceil (2011 - \sqrt{y})^2 \rceil, \ldots, \lfloor (2012 - \sqrt{y})^2 \rfloor$. The total number of ordered pairs is therefore given by

$$4023 + \sum_{n=0}^{2011^2} \left(\lfloor (2012 - \sqrt{n})^2 \rfloor - \lceil (2011 - \sqrt{n})^2 \rceil + 1 \right).$$

Typing this into Mathematica returns the result (after 18.6 minutes of CPU time[2]): $\boxed{5{,}425{,}870{,}956}$.

[2]Dual 2.33 GHz IntelCore2 Duo