Coffee Hour Problems of the Week

Edited by Matthew McMullen

Otterbein University

Spring 2012

Week 1. *Proposed by Matthew McMullen.*

Can you find two rational numbers whose sum and product are reciprocals of each other?

Week 2. *Proposed by Matthew McMullen.*

Find

$$\lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}\, dx.$$

Week 3. *Proposed by Matthew McMullen.*

You have a bag that contains two dice. One is a standard, fair die, and the other is a fair die, but the 1 has been replaced with a 6. You reach into the bag and choose one of the two dice at random, then you roll the die and get a 6. What is the probability that you'll get a six if you roll the same die again?

Week 4. *Proposed by Matthew McMullen.*

Two cubes have rational side-lengths. Their total volume is 19. Find both side-lengths.

Week 5. *Proposed by Matthew McMullen.*

For $a, b > -100$, define $a \oplus b$ to be the effective percent change of an $a\%$ change followed by a $b\%$ change. For example, $10 \oplus -15 = -6.5$, since 100 increased by 10% is 110, and 110 decreased by 15% is 93.5. Is the interval $(-100, \infty)$ a commutative group under $\oplus$?

Week 6. *Purdue University Problem of the Week (paraphrased).*

Let S be a finite set, and let $P(S)$ denote the power set of S. Suppose that $*$ is a function from $P(S)$ to itself. For every $A \in P(S)$, let A^* denote the image of A under $*$. If $A^* \subseteq B^*$ whenever $A \subseteq B \subseteq S$, show that $*$ has a fixed point. Does this conclusion necessarily hold if S is infinite?

Week 7. *Seen on a T-shirt.*

Let $\overline{AB}$ and $\overline{CD}$ be two chords of a circle. Suppose these chords are perpendicular and meet at the point P inside the circle. If $AP = 2$, $PD = 3$, and $PB = 6$, find the diameter of the circle.

Week 9. *Proposed by Duane Buck.*

Find the distance from the point (x, y) to the line segment with endpoints (x_1, y_1) and (x_2, y_2).

Week 10. *Proposed by Matthew McMullen.*

My 30-year mortgage has a fixed APR of 5.25% compounded monthly. In what month will my payment toward principal exceed my interest payment for the first time?

Week 11. *Proposed by Matthew McMullen.*

One prepayment strategy for installment loans is to pay one-and-a-half times the monthly payment each month. According to my realtor, this cuts the length of a 30-year mortgage roughly in half.

(a) For what fixed APR, compounded monthly, is this *exactly* true?

(b) At my fixed APR of 5.25%, compounded monthly, how long will it take me to pay off my 30-year mortgage using this strategy?

Week 12. *Proposed by Matthew McMullen.*

Find $p > 1$ such that

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 2012.$$

Week 13. *Purdue U. Problem of the Week.*

A tetrahedron has a base which is an equilateral triangle of edge length one. The other three faces are congruent isosceles triangles with one edge an edge of the base and the other edges of length three. Find the length of the shortest path, lying in the union of the three isosceles faces, which starts and ends at the same vertex of the base and which meets every line segment drawn from the top vertex to the perimeter of the base triangle.

Week 14. *Proposed by Matthew McMullen.*

Let $r, s > 0$. Find the parabola that opens down, goes through the points $(0, 0)$ and (r, s), and cuts off the minimum possible area in the first quadrant. What is this minimum area?

Week 15. *From* The College Mathematics Journal.

Let $ABCD$ be a square with sides of length 1. Suppose K and L are points on the sides BC and CD, respectively, so that the perimeter of triangle KCL is 2. If triangle AKL has minimum area, determine the measures of its angles.

Summer! *Classic unsolved problem from 2009.*

A lock consists of exactly n buttons. The lock is opened by pressing all of the buttons in some unknown order. The lock doesn't reset if the wrong order is tried. Assuming optimal strategy, what is the maximum number of button-presses that are needed to open the lock?