Coffee Hour Problems of the Week (solutions)

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Week 1. *Proposed by Matthew McMullen.*

Let $f(x)$ be a polynomial with leading coefficient 1 and degree 2011. Suppose, moreover, that the roots of f are the first 2011 positive integers. Find the constant term, the sum of all of the coefficients, the coefficient of the x^{2010} term, and the coefficient of the x^{2009} term.

Solution. We are given that $f(x) = \prod_{k=1}^{2011}(x-k)$. Thus, the constant term is given by $f(0) = \boxed{-2011!}$, and the sum of all of the coefficients is given by $f(1) = \boxed{0}$. It is also well-known that the coefficient of the second-highest power of x for any monic polynomial is the opposite of the sum of its roots. Therefore, the coefficient of the x^{2010} term is

$$-\sum_{k=1}^{2011} k = -\frac{(2011)(2012)}{2} = \boxed{-2,023,066}.$$

The coefficient of the x^{2009} term is given by

$$\begin{aligned}
\sum_{1\le k<m\le 2011} k\cdot m &= \sum_{k=1}^{2010}(k+1)(1+2+\cdots k) \\
&= \sum_{k=1}^{2010}\frac{k(k+1)^2}{2} \\
&= \frac{1}{2}\sum_{k=1}^{2010}k^3 + \sum_{k=1}^{2010}k^2 + \frac{1}{2}\sum_{k=1}^{2010}k \\
&= \frac{(2010\cdot 2011)^2}{8} + \frac{2010(2011)(2(2010)+1)}{6} + \frac{2010(2011)}{4} \\
&= \boxed{2{,}045{,}041{,}554{,}425}.
\end{aligned}$$

Week 2. *Proposed by Matthew McMullen.*

Is it possible for a rational function to cross its oblique asymptote? If not, prove it; if so, find an example of a rational function that crosses its oblique asymptote exactly 2011 times.

Solution. A rational function may cross its oblique asymptote any finite number of times. The rational function

$$R(x) = x + \frac{(x-1)(x-2)\cdots(x-2011)}{x^{2012}+1}$$

crosses its oblique asymptote, $y = x$, exactly 2011 times.

Week 3. *Proposed by Matthew McMullen.*

A game is played by repeatedly rolling one fair six-sided die until the number rolled is less than the number that was just rolled. What is the probability that you will roll the die more than four times?

Solution. Let $p(n)$ be the probability that you roll the die exactly n times. There are fifteen possible ways to roll the die exactly two times: 65, 64, 63, 62, 61, 54, 53, 52, 51, 43, 42, 41, 32, 31, and 21. Therefore, $p(2) = 15/36$. Similarly, there are 70 ways to roll the die exactly three times and 210 ways to roll the die exactly four times. Thus, $p(3) = 70/216$ and $p(4) = 210/1296$. The probability that you will roll the die more than four times is then

$$1 - \big(p(2) + p(3) + p(4)\big) = \boxed{\tfrac{7}{72}}.$$

Week 4. *Proposed by Matthew McMullen. (Part (a) suggested by Joshua Neiswanger.)*

(a) For each $a > 0$, consider the line perpendicular to the curve $y = x^2$ at $x = a$. This line intersects the curve at another point. Find the maximum possible value of the x-coordinate of this second intersection point.

(b) Same as **(a)**, but change the curve to $y = x^4$. (This one will require some outside help!)

Solution. **(a)** Since the slope of $y = x^2$ at a is $2a$, the slope of the perpendicular at this point is $-1/(2a)$. The equation of the perpendicular is then

$$y - a^2 = \frac{-1}{2a}(x - a).$$

Therefore, the x-coordinate of the second intersection point satisfies

$$x^2 - a^2 = \frac{-1}{2a}(x - a),$$

where $x \neq a$. Thus, $x = -a - 1/(2a)$. Using calculus, we see that this function attains its maximum value of $\boxed{-\sqrt{2}}$ when $a = 1/\sqrt{2}$.

(b) Similar to **(a)**, we see that the x-coordinate of the second intersection point satisfies

$$x^4 - a^4 = \frac{-1}{4a^3}(x - a),$$

where $x \neq a$. Thus,

$$x^3 + ax^2 + a^2x + a^3 = \frac{-1}{4a^3}. \tag{1}$$

If we differentiate this implicitly with respect to a and set the derivative equal to zero, we see that

$$12a^6 + 8a^5x + 4a^4x^2 = 3. \tag{2}$$

Using WolframAlpha to solve (1) and (2) simultaneously (for $a > 0$) yields $a \approx 0.844309$ and $x \approx \boxed{-1.06833}$.

Week 5. *Proposed by Matthew McMullen.*

You are trying to make a bank shot in a game of pool. The cue ball is a units from the rail you want to bank off of, the target ball is b units from that rail, and the two balls are d units apart.

(a) Assuming you don't put any spin on the ball, at which point on the rail should you aim the cue ball?

(b) You aim at the point found in **(a)**, but you are off by x units. Find an expression that gives the minimum distance between the cue ball and the target ball as a function of x.

Solution. **(a)** Let A and B be the points on the rail closest to the cue and target balls, respectively. Let $c = \sqrt{d^2 - (b - a)^2}$ (so c is the distance between

A and B). Let x be the distance between A and the point you need to aim at. By similar triangles, $x/a = (c - x)/b$. Thus, $x = \frac{ac}{a+b}$.

(b) Locate the cue and target balls in the plane; the cue ball at $(-ac/(a+b), a)$ and the target ball at $(bc/(a+b), b)$. If we aim at the point $(x_0, 0)$, then the slope of the "ricochet line" is

$$m = \frac{a}{x_0 + \frac{ac}{a+b}},$$

and the equation of this line is $y = mx - mx_0$. Then the distance between the target ball and this line is

$$\frac{|b - \frac{mbc}{a+b} + mx_0|}{\sqrt{m^2+1}} = \frac{|x_0|(a+b)^2}{\sqrt{a^2(a+b)^2 + (x_0(a+b)+ac)^2}}.$$

Therefore, the function we seek is

$$f(x) = \frac{|x|(a+b)^2}{\sqrt{a^2(a+b)^2 + (x(a+b)+ac)^2}},$$

where $0 < x < bc/(a+b)$ means we've aimed x units to the right of the target, and $-ac/(a+b) < x < 0$ means we've aimed $|x|$ units to the left of the target.

Week 6. *From Stewart's* Calculus.

$ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have.

Solution. Let $a \le 1$ be the distance from A to E and $b \le 1$ be the distance from A to F. Let (x, y) be the coordinates of A after the paper is folded. Then, by Pythagoras' Theorem, $(a-x)^2 + y^2 = a^2$ and $x^2 + (y-b)^2 = b^2$. Since $x^2 + y^2 = 1$, these two equations tell us that $a = 1/(2x)$ and $b = 1/(2y)$. Also, if $a = 1$, then $x = 1/2$; and, if $b = 1$, then $y = 1/2$ and $x = \sqrt{3}/2$.

Therefore, the function we wish to optimize is given by

$$A(x) = \frac{1}{2}ab = \frac{1}{2} \cdot \frac{1}{2x} \cdot \frac{1}{2y} = \frac{1}{8x\sqrt{1-x^2}},$$

where $1/2 \le x \le \sqrt{3}/2$. The critical point of this function is $x = \sqrt{2}/2$.

Testing the critical point and the endpoints, we see that the maximum possible area is $A(1/2) = A(\sqrt{3}/2) = \boxed{1/(2\sqrt{3})}$ and the minimum possible area is $A(\sqrt{2}/2) = \boxed{1/4}$.

Week 7. *Proposed by Matthew McMullen.*

There is a line through the origin that divides the region bounded by the parabola $y = 2011x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

Solution. Let m be the slope we're trying to find. We are given that

$$\int_0^a (2011x - x^2 - mx)\,dx = \frac{1}{2}\int_0^{2011} (2011x - x^2)\,dx, \tag{3}$$

where $a > 0$ is the x-coordinate of the intersection point of the parabola and the bisecting line. By equating the parabola and the line, we get $a = 2011 - m$.

After integrating, we see that the left-hand side of (3) is $a^3/6$ and the right-hand side of (3) is $2011^3/12$. Therefore, $a = 2011/\sqrt[3]{2}$; and so

$$\boxed{m = 2011 - \tfrac{2011}{\sqrt[3]{2}}}.$$

Week 8. *Proposed by Ryan Berndt and Matthew McMullen.*

Let n be a positive integer and $f \in C^n([0,1])$, with $f \geq 0$ and $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$. Let M_n be the maximum value of $|f^{(n)}(x)|$ on $[0,1]$. Show that

$$\int_0^1 f(x)\,dx \leq \frac{1}{n!}\int_0^1 |f^{(n)}(x)|\,dx \quad \text{and} \quad \int_0^1 f(x)\,dx \leq \frac{M_n}{(n+1)!}.$$

Solution. One can show, using induction and integration by parts, that

$$\int_0^1 f(x)\,dx = \int_0^1 f^{(k)}(x)\frac{(1-x)^k}{k!}\,dx,$$

for all $k \geq 0$. Therefore,

$$\int_0^1 f(x)\,dx \leq \frac{1}{n!}\int_0^1 |f^{(n)}(x)|(1-x)^n\,dx \leq \frac{1}{n!}\int_0^1 |f^{(n)}(x)|\,dx,$$

and

$$\int_0^1 f(x)\,dx \leq \frac{M_n}{n!}\int_0^1 (1-x)^n\,dx = \frac{M_n}{(n+1)!}.$$

Week 9. *From the 2010 Putnam Competition.*

Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

Solution. We can maximize k be minimizing the common sum of the numbers in each box. Since the number n must appear in a box, this common sum must be greater than or equal to n. For n odd, the common sum of n can be realized by letting the boxes be $\{n\}, \{1, n-1\}, \{2, n-2\}, \ldots, \{(n-1)/2, (n+1)/2\}$. For n even, the common sum of n cannot be realized, but $n+1$ can be by letting the boxes be $\{1, n\}, \{2, n-1\}, \{3, n-2\}, \ldots, \{n/2, n/2+1\}$.

Therefore, for n odd, the largest such k is $\boxed{(n+1)/2}$; and, for n even, the largest such k is $\boxed{n/2}$.

Week 10. *Proposed by Zeying Wang.*

(a) Is it possible to plant nine trees such that there are nine rows with three trees in each row? (The rows can be horizontal, vertical, and/or diagonal.)

(b) Is it possible to plant ten trees such that there are ten rows with three trees in each row? (The rows can be horizontal, vertical, and/or diagonal.)

Challenge. Try **(a)** and **(b)** with the added condition that each tree is a member of exactly three rows.

Solution. Both of these configurations are possible, even with the added condition given in the challenge. See the projective geometry theorems of Pappus and Desargues for details. (*Projective Geometry: From Foundations to Applications*, by A. Beutelspacher and U. Rosenbaum is a good source.)