Coffee Hour Problems of the Week (solutions)

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Week 1. *Proposed by Matthew McMullen.*

Let S be the set of all degree three monic polynomials with integer coefficients whose roots (both real and complex) all have modulus 2011. Find the number of elements in S.

Solution. Let $f \in S$. If f has only real roots, then there are exactly four possibilities for f: $(x-2011)^3$, $(x+2011)^3$, $(x-2011)(x+2011)^2$, and $(x+2011)(x-2011)^2$. Suppose f has a non-real root, say $a+bi$. Then, since non-real roots come in conjugate pairs, $a-bi$ is also a root. Notice that

$$(x-(a+bi))(x-(a-bi)) = x^2 - 2ax + a^2 + b^2 = x^2 - 2ax + 2011;$$

therefore, $2a$ must be an integer. The only possibilities for a are thus

$$0, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \ldots, \pm\frac{4021}{2}.$$

Each of these 8043 choices for a gives two different possibilities for f, namely $(x \pm 2011)(x^2 - 2ax + 2011)$. Therefore, S has $2(8043) + 4 = \boxed{16{,}090}$ elements.

Week 2. *Proposed by Matthew McMullen.*

A collection of distinct concentric circles is said to be in *integer standard position* if they have integer radii and their common center has integer coordinates. Is it possible to find a rectangle with integer length and width and four distinct concentric circles in integer standard position such that each vertex of the rectangle is on a different circle? If not, prove it; if so, find an explicit example.

Solution. It is possible to have the desired setup. The key is to find two distinct Pythagorean triples (not necessarily primitive) that share a leg; for example, $(9, 12, 15)$ and $(5, 12, 13)$. One (of infinitely many) explicit example is given by the circles centered at $(0, 9)$ with radii 5, 9, 13, and 15 and the rectangle with vertices $(0, 4)$, $(0, 0)$, $(12, 4)$, and $(12, 0)$.

Week 3. *Ancient result proposed by Matthew McMullen.*

Let $f(x)$ be a parabola and l a line that intersects f in two distinct places, say P_1 and P_2. Let T be the triangle whose vertices are P_1, P_2, and the point on f whose x-coordinate is the average of the x-coordinates of P_1 and P_2. Archimedes (and many others throughout history) proved that the area bounded by f and l is four thirds the area bounded by T. Can you prove this?

Solution. My proof is extremely inelegant and uses calculus; Archimedes' solution (see the *Quadrature of the Parabola*) is much better.

Without loss of generality, let $f(x) = ax^2$, where $a > 0$, and let $y = mx + b$ be the equation for l, where $m^2 + 4ab > 0$ (to ensure that there are two distinct intersection points). Let x_1 and x_2 denote the x-coordinates of P_1 and P_2, where $x_1 < x_2$. Then the area bounded by f and l is given by

$$\int_{x_1}^{x_2} (mx + b - ax^2)\, dx.$$

Using the fact that $ax_i^2 - mx_i - b = 0$ for $i = 1, 2$, and with our ultimate result in mind, we see that this integral is equal to

$$\frac{4}{3} \cdot \frac{1}{2}(x_2 - x_1)\sqrt{1+m^2} \cdot \frac{\frac{m^2}{4a} + b}{\sqrt{1+m^2}}. \tag{1}$$

To find the area bounded by T, notice that the distance between P_1 and P_2 is $(x_2 - x_1)\sqrt{1+m^2}$. Therefore, to prove Archimedes' result, it is sufficient to show that the height of T is equal to the last factor of (1). This height is the distance from the point $\big((x_1 + x_2)/2, a(x_1 + x_2)^2/4\big)$ to the line l, which can be shown to be equal to the required factor.

Week 4. *Proposed by Matthew McMullen.*

For n a positive integer, let $D(n)$ be the smallest positive integer with exactly n (positive) divisors.

(a) Find $D(n)$ for $n = 1, 2, \ldots, 10$.

(b) Find $D(2011)$ and $D(2010)$.

(c) Can you find a general formula for $D(n)$?

Solution. By counting the number of divisors for the numbers 1 through 64, we see that the first ten values of $D(n)$ (in order) are 1, 2, 4, 6, 16, 12, 64, 24, 36, and 48. Since 2011 is prime, the only numbers with exactly 2011 divisors are of the form p^{2010}, where p is prime. Therefore, $\boxed{D(2011) = 2^{2010}}$.

While a general formula for $D(n)$ is unknown, there is an algorithm to find it. We will use the fact that if $N = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ is the prime factorization of N, then N has exactly $(a_1+1)(a_2+1)\cdots(a_k+1)$ divisors. To find $D(n)$, we have to find the smallest $N = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ with $(a_1+1)(a_2+1)\cdots(a_k+1) = n$. First list all of the ways to write n as a product of non-increasing positive integers. Then, for each product in our list, we consider the number $\overline{N} = p_1^{b_1-1} p_2^{b_2-1} \cdots p_k^{b_k-1}$, where p_i denotes the ith prime and b_i denotes the ith factor in the product under consideration. The smallest such $\overline{N}$ is $D(n)$.

For $n = 2010$, we have 15 ways to write n as a product of non-increasing positive integers. For example, the products $67\cdot 30$ and $201\cdot 5\cdot 2$ give $\overline{N} = 2^{66}\cdot 3^{29}$ and $\overline{N} = 2^{200}\cdot 3^4 \cdot 5$, respectively. Comparing all 15 possible $\overline{N}$'s, we see that $\boxed{D(2010) = 2^{66}\cdot 3^4\cdot 5^2\cdot 7}$.

Week 5. *Proposed by Matthew McMullen.*

For n a positive integer, let

$$f(n) = \int_0^1 \frac{1}{x^{1/n} + x^{1/(n+1)}}\, dx.$$

(a) Show that $f(n)$ exists for all n.

(b) Find $f(1)$ and $f(2)$.

(c) Can you find a general formula for $f(n)$?

Solution. $f(n)$ exists, since, for all positive integers n,

$$f(n) < \int_0^1 \frac{1}{\sqrt{x}}\, dx = 2.$$

Using the substitution $u^2 = x$ gives

$$f(1) = 2\int_0^1 \frac{1}{u+1}\, du = \boxed{2\ln 2},$$

and using the substitution $u^6 = x$ gives

$$f(2) = 6\int_0^1 \frac{u^3}{u+1}\, du = 6\int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1}\right) du = \boxed{5 - 6\ln 2}.$$

In general, using the substitution $u^{n(n+1)} = x$, we get

$$\begin{aligned}
f(n) &= n(n+1)\int_0^1 \frac{u^{n^2-1}}{u+1}\,du \\
&= n(n+1)\int_0^1 \left(u^{n^2-2} - u^{n^2-3} + \cdots \pm 1 \mp \frac{1}{u+1}\right)du \\
&= n(n+1)\left(\frac{1}{n^2-1} - \frac{1}{n^2-2} + \cdots \pm 1 \mp \ln 2\right),
\end{aligned}$$

where we use the top signs if n is even and the bottom signs if n is odd. This can be written more concisely as

$$f(n) = (-1)^{n+1}\, n(n+1)\left(\ln 2 - \sum_{k=\lfloor \frac{n^2+1}{2} \rfloor}^{n^2-1} \frac{1}{k}\right).$$

Week 6. *Proposed by Ryan Berndt.*

Let $a \geq b \geq 0$. Show that

$$a^{1/n} - b^{1/n} \leq (a-b)^{1/n},$$

for all positive integers n.

Solution. Let n be a positive integer. Put $u = a^{1/n}$ and $v = b^{1/n}$. What we want to show is equivalent to showing that

$$(u-v)^n \leq u^n - v^n.$$

Since $a \geq b$, we have that $u \geq v$. Thus, $u = v + t$, for some $t \geq 0$. Therefore, we need to show that

$$t^n \leq (v+t)^n - v^n.$$

By the binomial theorem, the right-hand side of this inequality is

$$t^n + \sum_{k=1}^{n-1} \binom{n}{k} v^{n-k} t^k \geq t^n,$$

with equality if and only if either $n = 1$ or $v = 0$ (in which case $b = 0$) or $t = 0$ (in which case $a = b$).

Week 7. *Proposed by Matthew McMullen.*

Let C be the circle with radius 1 centered at $(0,1)$. Let C^* be C minus the point $(0,2)$.

(a) Show that the mapping

$$(a,b) \mapsto \frac{2a}{2-b}$$

gives a one-to-one correspondence between C^* and the set of all real numbers.

(b) Can you describe an explicit one-to-one correspondence between C and the set of all real numbers?

Solution. To show that the map is 1-1, suppose (a_1, b_1) and (a_2, b_2) are in C^* (so $a_i^2 = b_i(2-b_i)$ and $b_i \neq 2$ for $i = 1, 2$) and

$$\frac{2a_1}{2-b_1} = \frac{2a_2}{2-b_2}.$$

Squaring both sides and simplifying gives

$$b_1(2-b_2) = b_2(2-b_1).$$

Thus, $b_1 = b_2$; and hence, $a_1 = a_2$.

To show that the map is onto, let x be a real number and put $a = 4x/(x^2+4)$ and $b = 2x^2/(x^2+4)$. Then $b \neq 2$ and

$$b(2-b) = \frac{2x^2}{x^2+4} \cdot \frac{8}{x^2+4} = \frac{16x^2}{(x^2+4)^2} = a^2;$$

so (a,b) is in C^*. Also,

$$\frac{2a}{2-b} = \frac{\frac{8x}{x^2+4}}{\frac{8}{x^2+4}} = x.$$

Any 1-1 correspondence between all of C and the real line will be, in some sense, ugly. One idea is to cobble together several bijections. You can map C to $[0, 2\pi)$ by shifting C down 1 and using polar coordinates. You can map the *open* interval $(0, 2\pi)$ to C^* by using polar coordinates, rotating 90 degrees counterclockwise, and shifting up 1. The main problem, then, is to describe a bijection between $[0, 2\pi)$ and $(0, 2\pi)$.

One way to do this is to let $\{q_1, q_2, \dots\}$ be an enumeration of the rationals in $(0, 2\pi)$. Then define your map by $x \mapsto x$, for x irrational, $0 \mapsto q_1$, and $q_i \mapsto q_{i+1}$, for $i = 1, 2, \dots$.

Week 8. *Suggested by Molly Clairemont (from mathschallenge.net).*

Let b be a positive integer. Prove that the Diophantine equation

$$(x^2 + (b - x)y)^2 = 1$$

has at least four solutions over the positive integers.

Solution. We can rewrite the equation as the two equations

$$(x - b)y = x^2 \pm 1.$$

For $b = 1$, we clearly have infinitely many solutions: $(t, t + 1)$, for all positive integers t. If we make the substitution $u = x - b$ and solve for y, we see that either

$$y = u + 2b + \frac{b^2 + 1}{u} \quad \text{or} \quad y = u + 2b + \frac{b^2 - 1}{u}.$$

Thus, y is a positive integer for $u \in \{1, b^2 + 1, b + 1, b - 1, b^2 - 1\}$. If $b > 2$, these choices for u actually give us six distinct solutions:

$$(b + 1, b^2 + 2b + 2), (b^2 + b + 1, b^2 + 2b + 2), (b + 1, b^2 + 2b),$$

$$(2b + 1, 4b), (2b - 1, 4b), (b^2 + b - 1, b^2 + 2b).$$

For $b = 2$, we have four solutions: $(3, 10), (3, 8), (5, 8)$, and $(7, 10)$.

Week 9. *Proposed by Matthew McMullen, inspired by a problem suggested by Molly Clairemont.*

A certain dice game is played by starting with k fair dice, rolling them all at once, and removing any sixes that appear. This is called a trial. The next trial consists of rolling the remaining dice (or die) all at once and removing any sixes that appear. The trials continue in this manner until there are no dice left, at which point the game is over. For $k = 2$, find the expected number of trials per game.

Solution. Let $p(x)$ be the probability that the game ends after exactly x trials; so $p(1) = 1/36$. To find $p(x)$ for $x > 1$, notice that our first trial will either yield no sixes, with probability $25/36$, or exactly one six, with probability $2(1/6)(5/6)$. In the first case, the remaining $x - 1$ trials will have to eliminate both sixes. This occurs with probability $p(x - 1)$. In the second case, we need to get no sixes in the next $x - 2$ trials and then one six in the last trial. Therefore, for $x > 1$,

$$p(x) = \frac{25}{36}p(x - 1) + \frac{1}{18}\left(\frac{5}{6}\right)^{x-1}.$$

To find the expected value of x, we multiply through by $36x$ and do some clever rearranging to get

$$\begin{aligned}11\sum_{x=1}^{\infty} x\,p(x) &= 11/36 + 25\sum_{x=2}^{\infty}[(x-1)\,p(x-1) - x\,p(x)] + 25\sum_{x=2}^{\infty} p(x-1) + 2\sum_{x=2}^{\infty} x(5/6)^{x-1} \\ &= 11/36 + 25(1/36) + 25(1) + 2(35) \\ &= 96.\end{aligned}$$

Therefore, the expected number of trials per game[1] is $\boxed{96/11}$.

Week 10. (*Continuation of last week's problem.*)

The decaying dice game from Week 9 is played starting with k dice. Let $p_k(x)$ be the probability that the game ends after exactly x trials. Can you find an equation for $p_k(x)$? Can you find the expected value and standard deviation of x?

Solution. Obviously, $p_k(1) = (1/6)^k$. If we get exactly $0 \le i < k$ sixes on the first roll (an event which follows a binomial distribution), then we need to get rid of the remaining $k-i$ sixes in the last $x-1$ rolls. Therefore, for $x > 1$,

$$p_k(x) = \sum_{i=0}^{k-1} \binom{k}{i} \left(\frac{5}{6}\right)^{k-i} \left(\frac{1}{6}\right)^{i} p_{k-i}(x-1).$$

Let μ_k be the expected value of x, and let σ_k^2 be the variance of x. Then, similar to the solution in Week 9, one can show[2] that

$$\mu_k = \frac{1 + \sum_{i=1}^{k-1} \binom{k}{i} \left(\frac{5}{6}\right)^{k-i} \left(\frac{1}{6}\right)^{i} \mu_{k-i}}{1 - \left(\frac{5}{6}\right)^{k}}$$

and

$$\sigma_k^2 = \frac{(1-\mu_k)^2 \left(\frac{1}{6}\right)^k + \left(\frac{5}{6}\right)^k + \sum_{i=1}^{k-1} \binom{k}{i} \left(\frac{5}{6}\right)^{k-i} \left(\frac{1}{6}\right)^{i} (\sigma_{k-i}^2 + (\mu_k - \mu_{k-i} - 1)^2)}{1 - \left(\frac{5}{6}\right)^{k}}.$$

[1] In our above calculations, we used the fact that $p(x)$ is a probability distribution, so its values sum to 1. Also, it can be shown that $\lim_{x\to\infty} x\,p(x) = 0$.

[2] Trust me!