Coffee Hour Problems of the Week (solutions)

Edited by Matthew McMullen

Otterbein University

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Week 1. *Proposed by Matthew McMullen.*

A gambling game is played by rolling three fair dice. If the numbers that are rolled can be the side lengths of a triangle, then you win \$1; otherwise, you lose \$1. Would you play this game?

Solution. There are 6^3 equally-likely simple events in our sample space. We will count the number of these that can be the side lengths of a triangle. We will use the fact that three positive numbers can be the side lengths of a triangle if and only if the sum of the two smaller numbers is greater than the third number.

Let $a \le b \le c$ denote the three numbers rolled. If $c = 1$, then $a = b = 1$ and this is a winning triple. If $c = 2$ and $b = 2$, then either $a = 2$ or $a = 1$ gives a winning triple. Notice, however, that there are three ways to roll two 2s and a 1. (Also notice that $c = 2$ and $b = 1$ forces $a = 1$, which does not produce a winning triple.)

We will leave off the details, but we can continue in this way, fixing c and then testing the different options for a and b and seeing which ones yield a winning triple (remembering that order matters when counting simple events). In summary, 1 winning triple has $c = 1$, 4 winning triples have $c = 2$, 10 winning triples have $c = 3$, 19 winning triples have $c = 4$, 31 winning triples have $c = 5$, and 46 winning triples have $c = 6$. So the probability of winning this game is $111/216$ or $\boxed{37/72}$. Since this is slightly greater than one half, I would play this game (but with an expected win of less than 2.8¢ per bet, it might take a while to see any results!).

Week 2. *Proposed by Matthew McMullen.*

(a) Factor $x^4 + 2500$ as much as possible (over the integers).

(b) Find all real numbers α such that $x^4 + \alpha$ is factorable (over the integers).

Solution. We will tackle (b) first. The first possibility is that $x^4 + \alpha$ has a linear factor. This will occur if and only if it has an integer root. Thus, $x^4 + \alpha$ has a linear factor if and only if $\alpha = -k^4$, for some integer k.

The second possibility is that

$$x^4 + \alpha = (x^2 + ax + b)(x^2 + cx + d),$$

for some integers a, b, c, d. By equating the coefficients of x^3, we see that $c = -a$. Then, by equating the other coefficients, we have that $b+d-a^2 = 0$, $a(d-b) = 0$, and $bd = \alpha$.

If $a = 0$, then $d = -b$ and $\boxed{\alpha = -b^2}$. (This actually subsumes our first possibility above.) If $b = d$, then some algebraic manipulation shows us that $4\alpha = a^4$. Thus, $\boxed{\alpha = 4k^4}$ for some integer k. In this case, our factorization becomes

$$x^4 + 4k^4 = (x^2 + 2kx + 2k^2)(x^2 - 2kx + 2k^2).$$

When $\alpha = 2500 = 4 \cdot 5^4$ we have

$$\boxed{x^4 + 2500 = (x^2 + 10x + 50)(x^2 - 10x + 50)}.$$

Week 3. *Proposed by Matthew McMullen.*

(a) Find the last digit of the 123456789th Fibonacci number.

(b) Let F_n denote the nth Fibonacci number. Prove that

$$F_n = \sum_{k=1}^{\infty} \binom{n-k}{k-1}.$$

(We define $\binom{n}{k} = 0$ for $k > n$.)

Solution. (a) If we start writing down the last digits of the Fibonacci numbers, we notice that they repeat in cycles of length sixty. Since $123456789 = 60(2057613) + 9$, the last digit of the 123456789th Fibonacci number is the last digit of the 9th Fibonacci number, or $\boxed{4}$.

(b) The series represents the sums of the "shallow diagonals" of Pascal's Triangle. Given how this triangle is formed, it is fairly easy to see that the above equality is true. More rigorously, we use the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

where $0 \le k \le n$ (and we define $\binom{n-1}{-1} = 0 = \binom{n-1}{n}$).

Then

$$F_1 = 1 = \sum_{k=1}^{\infty} \binom{1-k}{k-1} \text{ and } F_2 = 1 = \sum_{k=1}^{\infty} \binom{2-k}{k-1},$$

and, for $n \geq 1$,

$$\begin{aligned} \sum_{k=1}^{\infty} \binom{n+2-k}{k-1} &= \sum_{k=1}^{\infty} \binom{n+1-k}{k-2} + \sum_{k=1}^{\infty} \binom{n+1-k}{k-1} \\ &= \sum_{k=1}^{\infty} \binom{n-k}{k-1} + \sum_{k=1}^{\infty} \binom{n+1-k}{k-1}. \end{aligned}$$

This proves our equality.

Week 4. *Proposed by Matthew McMullen.*

Let u, v, and w be the roots of $x^3 - 2010x + 2011$. Find, with proof, $\arctan u + \arctan v + \arctan w$.

Solution. We are given that

$$\begin{aligned} x^3 - 2010x + 2011 &= (x-u)(x-v)(x-w) \\ &= x^3 - (u+v+w)x^2 + (uv+uw+vw)x - uvw. \end{aligned}$$

Using the sum formula for tangent and some algebraic manipulation, we see that

$$\begin{aligned} \tan(\arctan u + \arctan v + \arctan w) &= \frac{u+v+w-uvw}{1-(uv+uw+vw)} \\ &= \frac{2011}{1+2010} \\ &= 1. \end{aligned}$$

Therefore, $\arctan u + \arctan v + \arctan w = \pi/4 + \pi k$, for some integer k.

It is easy to see that two of the roots are positive, say u and v, and the third is negative. Therefore,

$$0 < \arctan u,\ \arctan v < \pi/2 \ \text{ and } \ -\pi/2 < \arctan w < 0;$$

and so,

$$-\pi/2 < \arctan u + \arctan v + \arctan w < \pi.$$

Thus, $\arctan u + \arctan v + \arctan w = \boxed{\pi/4}$.

Week 5. *Proposed by Matthew McMullen.*

(a) Let n be a nonnegative integer. Show that

$$\int_0^1 (x \ln x)^n \, dx = \frac{(-1)^n \, n!}{(n+1)^{n+1}}.$$

(b) Prove that

$$\int_0^1 x^x \, dx = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots .$$

Solution. It is fairly easy to see, using integration by parts and induction, that

$$\int_0^1 x^n (\ln x)^k \, dx = \frac{(-1)^k \, k!}{(n+1)^{k+1}},$$

for all nonnegative integers k. Putting $k = n$ solves (a).

For (b), we have

$$\begin{aligned}
\int_0^1 x^x \, dx &= \int_0^1 e^{x \ln x} \, dx \\
&= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!} \right) dx \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \int_0^1 (x \ln x)^n \, dx \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} \\
&= \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots .
\end{aligned}$$

Week 6. *Proposed by Matthew McMullen.*

(a) What is the probability that the determinant of a random 2×2 matrix with integer entries is even?

(b) Let p be prime. What is the probability that the determinant of a random 2×2 matrix with integer entries is divisible by p?

Solution. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the determinant of M is divisible by p if and only if $ad \equiv bc \pmod{p}$. Let $P(k)$ be the probability that the product of two randomly chosen integers is congruent to $k \pmod{p}$. Then

$$P(0) = \frac{1}{p} + \frac{1}{p} - \frac{1}{p^2} = \frac{2p-1}{p^2} \quad \text{and} \quad P(i) = \frac{p-1}{p^2},$$

for $i = 1, 2, \ldots, p-1$.

Thus, the probability that $ad \equiv bc \equiv 0 \pmod{p}$ is $(2p-1)^2/p^4$, and the probability that $ad \equiv bc \equiv i \pmod{p}$ is $(p-1)^2/p^4$, for $i = 1, 2, \ldots, p-1$. Therefore, the probability that the determinant of M is divisible by p is

$$\frac{(2p-1)^2}{p^4} + \frac{(p-1)^3}{p^4} = \frac{p^2+p-1}{p^3}.$$

Substituting $p = 2$, we see that the answer to (a) is $\boxed{5/8}$.

Week 7. *Proposed by Matthew McMullen.*

For each positive integer k, let $M(k)$ be the maximum value of $\prod_{j=1}^{k} x_j$, where $\sum_{j=1}^{k} x_j = 2010$ and each of the x_i's are positive. Find the maximum value of $M(k)$.

Solution. The Arithmetic/Geometric Mean Inequality tells us that

$$\sqrt[k]{x_1 \cdots x_k} \le \frac{x_1 + \cdots + x_k}{k},$$

with equality iff all of the x_i's are equal. Thus,

$$\prod_{j=1}^{k} x_j \le \left(\frac{1}{k} \sum_{j=1}^{k} x_j \right)^k = \left(\frac{2010}{k} \right)^k,$$

with equality iff $x_i = 2010/k$, for all i. Therefore,

$$M(k) = \left(\frac{2010}{k} \right)^k = e^{k \ln\left(\frac{2010}{k}\right)}.$$

To find the maximum value of $M(k)$, put $f(x) = x \ln(2010/x)$. Using calculus, we find that $f(x)$ attains its maximum value when $x = 2010/e \approx 739.4$. Since $f(739) > f(740)$, the maximum value of $M(k)$ is $\boxed{(2010/739)^{739}}$.

Week 8. *Proposed by Matthew McMullen.*

Let A and B be two points in the plane. Describe the set of all points that are twice as far from A as from B.

Solution. First suppose that $B = (0, 0)$ and $A = (d, 0)$, where $d > 0$. Let (x, y) be twice as far from A as from B. Then

$$2\sqrt{x^2 + y^2} = \sqrt{(x - d)^2 + y^2}.$$

After squaring both sides and completing the square for both x and y, we see that

$$\left(x + \frac{d}{3}\right)^2 + y^2 = \left(\frac{2d}{3}\right)^2.$$

Therefore, the set of all points that are twice as far from A as from B is, in this case, the circle of radius $2d/3$ centered at $(-d/3, 0)$.

In general, let d be the distance between A and B, and let l be the line through A and B. Let P be the point on l that is $d/3$ units "behind" B, relative to A. Then the set of all points that are twice as far from A as from B is the circle of radius $2d/3$ centered at P.

Week 9. *Proposed by Matthew McMullen.*

Find the last three digits of

$$\sum_{n=1}^{2010} n!\,.$$

Solution. Notice that 1000 divides $k!$ for $k \geq 15$; therefore, the last three digits of $\sum_{n=1}^{2010} n!$ are the last three digits of $\sum_{n=1}^{14} n!$. Thus,

$$\begin{aligned}\sum_{n=1}^{2010} n! &\equiv \sum_{n=1}^{14} n! \pmod{1000}\\ &\equiv 1 + 2 + 6 + 24 + 120 + 720 + 40 + 320\\ &\quad + 880 + 800 + 800 + 600 + 800 + 200 \pmod{1000}\\ &\equiv \boxed{313} \pmod{1000}.\end{aligned}$$

Week 10. *Proposed by Matthew McMullen.*

Suppose that $\log_a x = \log a^x$, for $a, x > 0$ and $a \neq 1$. Find the maximum possible value of a. (Here, log denotes the common logarithm.)

Solution. By the change of base formula and properties of logarithms, we are given that

$$\frac{\log x}{\log a} = x \log a,$$

or $(\log x)/x = (\log a)^2$. Using calculus, we can see that the maximum value of $(\log x)/x$ occurs when $x = e$. Thus the maximum value of a is

$$\boxed{10^{\sqrt{\frac{\log e}{e}}}} \approx 2.510.$$