Coffee Hour Problems of the Week

Edited by Matthew McMullen

Otterbein University

Winter 2011

Week 1. *Proposed by Matthew McMullen.*

Let $f(x)$ be a polynomial with leading coefficient 1 and degree 2011. Suppose, moreover, that the roots of f are the first 2011 positive integers. Find the constant term, the sum of all of the coefficients, the coefficient of the x^{2010} term, and the coefficient of the x^{2009} term.

Week 2. *Proposed by Matthew McMullen.*

Is it possible for a rational function to cross its oblique asymptote? If not, prove it; if so, find an example of a rational function that crosses its oblique asymptote exactly 2011 times.

Week 3. *Proposed by Matthew McMullen.*

A game is played by repeatedly rolling one fair six-sided die until the number rolled is less than the number that was just rolled. What is the probability that you will roll the die more than four times?

Week 4. *Proposed by Matthew McMullen. (Part (a) suggested by Joshua Neiswanger.)*

(a) For each $a > 0$, consider the line perpendicular to the curve $y = x^2$ at $x = a$. This line intersects the curve at another point. Find the maximum possible value of the x-coordinate of this second intersection point.

(b) Same as **(a)**, but change the curve to $y = x^4$. (This one will require some outside help!)

Week 5. *Proposed by Matthew McMullen.*

You are trying to make a bank shot in a game of pool. The cue ball is a units from the rail you want to bank off of, the target ball is b units from that rail, and the two balls are d units apart.

(a) Assuming you don't put any spin on the ball, at which point on the rail should you aim the cue ball?

(b) You aim at the point found in **(a)**, but you are off by x units. Find an expression that gives the minimum distance between the cue ball and the target ball as a function of x.

Week 6. *From Stewart's* Calculus.

$ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have.

Week 7. *Proposed by Matthew McMullen.*

There is a line through the origin that divides the region bounded by the parabola $y = 2011x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

Week 8. *Proposed by Ryan Berndt and Matthew McMullen.*

Let n be a positive integer and $f \in C^n([0,1])$, with $f \geq 0$ and $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$. Let M_n be the maximum value of $|f^{(n)}(x)|$ on $[0,1]$. Show that

$$\int_0^1 f(x)\,dx \leq \frac{1}{n!}\int_0^1 |f^{(n)}(x)|\,dx \quad \text{and} \quad \int_0^1 f(x)\,dx \leq \frac{M_n}{(n+1)!}.$$

Week 9. *From the 2010 Putnam Competition.*

Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1,2,3,6\}, \{4,8\}, \{5,7\}$ shows that the largest k is *at least* 3.]

Week 10. *Proposed by Zeying Wang.*

(a) Is it possible to plant nine trees such that there are nine rows with three trees in each row? (The rows can be horizontal, vertical, and/or diagonal.)

(b) Is it possible to plant ten trees such that there are ten rows with three trees in each row? (The rows can be horizontal, vertical, and/or diagonal.)

Challenge. Try **(a)** and **(b)** with the added condition that each tree is a member of exactly three rows.