Coffee Hour Problems of the Week

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Otterbein University

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Week 1. *Proposed by Matthew McMullen.*

Let S be the set of all degree three monic polynomials with integer coefficients whose roots (both real and complex) all have modulus 2011. Find the number of elements in S.

Week 2. *Proposed by Matthew McMullen.*

A collection of distinct concentric circles is said to be in *integer standard position* if they have integer radii and their common center has integer coordinates. Is it possible to find a rectangle with integer length and width and four distinct concentric circles in integer standard position such that each vertex of the rectangle is on a different circle? If not, prove it; if so, find an explicit example.

Week 3. *Ancient result proposed by Matthew McMullen.*

Let $f(x)$ be a parabola and l a line that intersects f in two distinct places, say P_1 and P_2. Let T be the triangle whose vertices are P_1, P_2, and the point on f whose x-coordinate is the average of the x-coordinates of P_1 and P_2. Archimedes (and many others throughout history) proved that the area bounded by f and l is four thirds the area bounded by T. Can you prove this?

Week 4. *Proposed by Matthew McMullen.*

For n a positive integer, let $D(n)$ be the smallest positive integer with exactly n (positive) divisors.

(a) Find $D(n)$ for $n = 1, 2, \ldots, 10$.

(b) Find $D(2011)$ and $D(2010)$.

(c) Can you find a general formula for $D(n)$?

Week 5. *Proposed by Matthew McMullen.*

For n a positive integer, let

$$f(n) = \int_0^1 \frac{1}{x^{1/n} + x^{1/(n+1)}}\, dx.$$

(a) Show that $f(n)$ exists for all n.

(b) Find $f(1)$ and $f(2)$.

(c) Can you find a general formula for $f(n)$?

Week 6. *Proposed by Ryan Berndt.*

Let $a \geq b \geq 0$. Show that

$$a^{1/n} - b^{1/n} \leq (a - b)^{1/n},$$

for all positive integers n.

Week 7. *Proposed by Matthew McMullen.*

Let C be the circle with radius 1 centered at $(0, 1)$. Let C^* be C minus the point $(0, 2)$.

(a) Show that the mapping
$$(a, b) \mapsto \frac{2a}{2-b}$$
gives a one-to-one correspondence between C^* and the set of all real numbers.

(b) Can you describe an explicit one-to-one correspondence between C and the set of all real numbers?

Week 8. *Suggested by Molly Clairemont (from mathschallenge.net).*

Let b be a positive integer. Prove that the Diophantine equation
$$(x^2 + (b-x)y)^2 = 1$$
has at least four solutions over the positive integers.

Week 9. *Proposed by Matthew McMullen, inspired by a problem suggested by Molly Clairemont.*

A certain dice game is played by starting with k fair dice, rolling them all at once, and removing any sixes that appear. This is called a trial. The next trial consists of rolling the remaining dice (or die) all at once and removing any sixes that appear. The trials continue in this manner until there are no dice left, at which point the game is over. For $k = 2$, find the expected number of trials per game.

Week 10. (*Continuation of last week's problem.*)

The decaying dice game from Week 9 is played starting with k dice. Let $p_k(x)$ be the probability that the game ends after exactly x trials. Can you find an equation for $p_k(x)$? Can you find the expected value and standard deviation of x?