Coffee Hour Problems of the Week

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Week 1. *Proposed by Matthew McMullen.*

A gambling game is played by rolling three fair dice. If the numbers that are rolled can be the side lengths of a triangle, then you win \$1; otherwise, you lose \$1. Would you play this game?

Week 2. *Proposed by Matthew McMullen.*

(a) Factor $x^4 + 2500$ as much as possible (over the integers).

(b) Find all real numbers α such that $x^4 + \alpha$ is factorable (over the integers).

Week 3. *Proposed by Matthew McMullen.*

(a) Find the last digit of the 123456789th Fibonacci number.

(b) Let F_n denote the nth Fibonacci number. Prove that

$$F_n = \sum_{k=1}^{\infty} \binom{n-k}{k-1}.$$

(We define $\binom{n}{k} = 0$ for $k > n$.)

Week 4. *Proposed by Matthew McMullen.*

Let u, v, and w be the roots of $x^3 - 2010x + 2011$. Find, with proof, $\arctan u + \arctan v + \arctan w$.

Week 5. *Proposed by Matthew McMullen.*

(a) Let n be a nonnegative integer. Show that

$$\int_0^1 (x \ln x)^n \, dx = \frac{(-1)^n \, n!}{(n+1)^{n+1}}.$$

(b) Prove that

$$\int_0^1 x^x \, dx = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots .$$

Week 6. *Proposed by Matthew McMullen.*

(a) What is the probability that the determinant of a random 2×2 matrix with integer entries is even?

(b) Let p be prime. What is the probability that the determinant of a random 2×2 matrix with integer entries is divisible by p?

Week 7. *Proposed by Matthew McMullen.*

For each positive integer k, let $M(k)$ be the maximum value of $\prod_{j=1}^k x_j$, where $\sum_{j=1}^k x_j = 2010$ and each of the x_i's are positive. Find the maximum value of $M(k)$.

Week 8. *Proposed by Matthew McMullen.*

Let A and B be two points in the plane. Describe the set of all points that are twice as far from A as from B.

Week 9. *Proposed by Matthew McMullen.*

Find the last three digits of

$$\sum_{n=1}^{2010} n!\,.$$

Week 10. *Proposed by Matthew McMullen.*

Suppose that $\log_a x = \log a^x$, for $a, x > 0$ and $a \neq 1$. Find the maximum possible value of a. (Here, log denotes the common logarithm.)