Coffee Hour Problems of the Week

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Winter 2010

Week 1. *Proposed by Matthew McMullen.*

Show that $\sqrt{2009}+\sqrt{2010}$ is a root of a fourth degree polynomial with integer coefficients. Is there a non-zero polynomial with integer coefficients and of degree *less than* four that $\sqrt{2009}+\sqrt{2010}$ is a root of?

Solution. Let $N=\sqrt{2009}+\sqrt{2010}$. Then

$$N^2 = 4019 + 2\sqrt{2009\cdot 2010}.$$

Subtracting 4019 from both sides and squaring both sides of the resulting equation yields

$$N^4 - 8038N^2 + 4019^2 = 4\cdot 2009\cdot 2010,$$

or

$$N^4 - 8038N^2 + 1 = 0.$$

Thus, N is a root of $x^4 - 8038x^2 + 1$.

For the second part of the problem, suppose that $aN^3 + bN^2 + cN + d = 0$, for some integers a, b, c, and d. If we multiply this out and collect like terms, we have

$$(8039a + c)\sqrt{2009} + (8037a + c)\sqrt{2010} + 2b\sqrt{2009\cdot 2010} + (4019b + d) = 0.$$

Since none of the above terms are like radicals, the only way this equation can be satisfied is if each of the "coefficients" is zero.[1] Therefore, $a=b=c=d=0$. No such lower-degree polynomial exists.

[1] We are hand-waving here. Can you prove this fact?

Week 2. *Proposed by Matthew McMullen.*

(a) Let $m \geq n \geq 0$. Show that

$$\sum_{i=n}^{m} \binom{i}{n} = \binom{m+1}{n+1}.$$

(b) Find

$$\sum_{i=1}^{1729} \binom{3739-i}{2010} \quad \text{and} \quad \sum_{i=1}^{1729} i \binom{3739-i}{2010}.$$

Solution. We will solve **(a)** using induction on m. For $m = n$, we have

$$\sum_{i=n}^{m} \binom{i}{n} = \binom{n}{n} = 1 = \binom{n+1}{n+1} = \binom{m+1}{n+1}.$$

Now suppose that the equality is true for some $k \geq n$. Then

$$\begin{aligned}
\sum_{i=n}^{k+1} \binom{i}{n} &= \sum_{i=n}^{k} \binom{i}{n} + \binom{k+1}{n} \\
&= \binom{k+1}{n+1} + \binom{k+1}{n} \\
&= \binom{k+2}{n+1},
\end{aligned}$$

by the induction hypothesis and Pascal's identity.

To evaluate the first sum in **(b)**, we write it "backwards" and use **(a)**:

$$\sum_{i=1}^{1729} \binom{3739-i}{2010} = \sum_{i=2010}^{3738} \binom{i}{2010} = \binom{3739}{2011}.$$

To evaluate the second sum, we write it as a double sum and apply **(a)** twice:

$$\sum_{i=1}^{1079} i \binom{3739-i}{2010} = \sum_{j=2011}^{3739} \left[\sum_{i=2010}^{j-1} \binom{i}{2010} \right] = \sum_{j=2011}^{3739} \binom{j}{2011} = \binom{3740}{2012}.$$

Week 3. *Proposed by Matthew McMullen.*

If N is a positive integer with at least two prime divisors, define the *delta value* of N to be $p-q$, where $p>q$ are the two largest prime divisors of N. Find the previous five years and the next five years with the same delta value as 2010.

Solution. Since $2010 = 2\cdot 3\cdot 5\cdot 67$, the delta value of 2010 is 62. The only prime pairs (q,p) with $p-q=62$ and $p\cdot q<2010$ are $(5,67), (11,73),$ and $(17,79)$. Going through the appropriate multiples of $p\cdot q$ in each case, we find that the five such previous years are $5^2\cdot 67 = 1675, 2\cdot 11\cdot 73 = 1606, 17\cdot 79 = 1343, 2^2\cdot 5\cdot 67 = 1340,$ and $3\cdot 5\cdot 67 = 1005$.

Similarly, we find that the next five such years are $3\cdot 11\cdot 73 = 2409, 2^3\cdot 5\cdot 67 = 2680, 2\cdot 17\cdot 79 = 2686, 3^2\cdot 5\cdot 67 = 3015,$ and $2^2\cdot 11\cdot 73 = 3212$.

Week 4. *Proposed by Matthew McMullen.*

Let $F_1 = 1 = F_2$ and $F_n = F_{n-2}+F_{n-1}$ for $n\geq 3$. (So F_n is the nth Fibonacci number.) Find all n such that $F_n = n^2$.

Solution. Notice that $n^2+(n+1)^2 > (n+2)^2$ for $n>3$. So, if $k>3$, $F_k>k^2$, and $F_{k+1} > (k+1)^2$, then

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} \\ &> k^2 + (k+1)^2 \\ &> (k+2)^2. \end{aligned}$$

Therefore, since $F_{13} = 233 > 13^2$ and $F_{14} = 377 > 14^2$, (strong) induction tells us that $F_n > n^2$ for $n\geq 13$. Looking at F_1 through F_{12}, we see that $F_1 = 1^2$ and $F_{12} = 12^2$. Therefore, $n=1$ and $n=12$ are the only solutions to $F_n = n^2$.

Week 5. *Proposed by Matthew McMullen.*

(a) When I type i^i into my TI-83 calculator, it gives me 0.2078795764. When I type in $(-i)^i$, it gives me 4.810477381. What are the exact values of these numbers? More generally, how would you "make sense" of z^w, where z and w are complex numbers (and $z\neq 0$)?

(b) Classify all complex numbers z and w with $|z|=1$ and $z^w\in\mathbb{R}$.

Solution. Recall Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

We may therefore write $i = e^{i(\pi/2+2\pi k)}$, where k is an integer. If we choose $k = 0$ (and if the "usual" rules of exponents apply), we have

$$i^i = (e^{\pi/2 i})^i = e^{-\pi/2} = 0.2078795764\ldots.$$

Similarly,

$$(-i)^i = (e^{-\pi/2 i})^i = e^{\pi/2} = 4.810477381\ldots.$$

In general, z^w is a multi-valued function which depends on our representation of z. We are leaving off many details, but if we restrict ourselves to writing $w = a + bi$ and $z = re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$ (which my calculator seems to be doing), and if we want all of the usual rules of exponents to apply, we get

$$\begin{aligned}
z^w &= e^{w \log z} \\
&= e^{(a+bi)\log(re^{i\theta})} \\
&= e^{(a+bi)(\log r + i\theta)} \\
&= e^{a\log r - b\theta} e^{(b\log r + a\theta)i}.
\end{aligned}$$

If $|z| = 1$, then $r = 1$; and so

$$z^w = e^{-b\theta} e^{a\theta i}.$$

The only way this can be real is if $a\theta$ is an integer multiple of π.

Week 6. *Proposed by Matthew McMullen.*

Suppose

$$\int_0^a \frac{1}{\sqrt{1+\sqrt{x}}}\, dx = 2010.$$

Find, with minimal computational aid, the first two digits of a.

Solution. The substitution $u = \sqrt{1+\sqrt{x}}$ transforms the above equation into

$$\int_1^w (u^2 - 1)\, du = \frac{1005}{2},$$

where $w = \sqrt{1+\sqrt{a}}$. Integrating the left-hand side and rewriting the equation yields

$$2w^3 - 6w - 3011 = 0.$$

After some calculation (which could, if need be, be done by hand), we see that $11.5 < w < 11.6$. Solving this for a gives us

$$17,226 < a < 17,839.$$

So the first two digits of a are 17.

Week 7. *2009 Ohio MAA Student Team Competition.*

Let P be a point picked at random inside the equilateral triangle ABC. What is the probability that the angle $\angle APB$ is an acute angle?

Solution. Without loss of generality, assume that A is the point $(-1,0)$, B is the point $(1,0)$, and C is the point $(0,\sqrt{3})$. Then $\angle APB$ is acute precisely when P is outside of the unit circle (and inside the triangle).

Let R denote the region inside the triangle and outside the unit circle, let l_1 be the line segment with endpoints $(0,0)$ and $(1/2,\sqrt{3}/2)$, and let l_2 be the line segment with endpoints $(0,0)$ and $(-1/2,\sqrt{3}/2)$. Then the area of R is the area of the triangle, minus the areas of two equilateral triangles with sides of length 1, minus the circular segment bounded by l_1 and l_2. Thus, the area of R is

$$\sqrt{3} - \frac{\sqrt{3}}{2} - \frac{\pi}{6},$$

and the probability we seek is

$$\frac{\sqrt{3} - \frac{\sqrt{3}}{2} - \frac{\pi}{6}}{\sqrt{3}} = \frac{3\sqrt{3} - \pi}{6\sqrt{3}} \approx 0.1977.$$

Week 8. *Proposed by Matthew McMullen.*

Find

$$\sum_{n=1}^{\infty} \frac{2n-1}{(4n-1)!}.$$

Solution. Using Taylor series, we have

$$\begin{aligned}
\sin 1 &= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots, \\
-\cos 1 &= -1 + \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \cdots, \text{ and} \\
e^{-1} &= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots.
\end{aligned}$$

Adding these three equations gives us

$$\begin{aligned}
\sin 1 - \cos 1 + e^{-1} &= \left(\frac{2}{2!} - \frac{2}{3!}\right) + \left(\frac{2}{6!} - \frac{2}{7!}\right) + \left(\frac{2}{10!} - \frac{2}{11!}\right) + \left(\frac{2}{14!} - \frac{2}{15!}\right) + \cdots \\
&= \frac{4}{3!} + \frac{12}{7!} + \frac{20}{11!} + \frac{28}{15!} + \cdots \\
&= 4\sum_{n=1}^{\infty} \frac{2n-1}{(4n-1)!}.
\end{aligned}$$

Thus, the answer is

$$\frac{\sin 1 - \cos 1 + e^{-1}}{4}.$$

Week 9. *Proposed by Matthew McMullen.*

Let T be the region bounded by an isosceles triangle. Mathematically describe all ways of dividing T into two equal-area pieces using a straight line.

Solution (outline). Suppose the triangle has vertices $(0,0)$, (b,h), and $(-b,h)$, where $b,h > 0$. Let P be the point (α, h), where $0 \le \alpha \le b$. We will find the point Q on the triangle such that $\overline{PQ}$ divides T as required. Since the line $x = 0$ divides T in half, we need to find Q such that the two triangular regions formed by $\overline{PQ}$ and $x = 0$ have equal areas. After much (messy) work, we find that

$$Q = \left(\frac{-\alpha b}{\alpha + b}, \frac{\alpha h}{\alpha + b}\right).$$

Since the line $y = h/\sqrt{2}$ also divides T in half, another possibility is that P has coordinates $(\alpha, h\alpha/b)$, where $b/\sqrt{2} \le \alpha \le b$. Then Q is the point on the triangle such that the two triangular regions formed by $\overline{PQ}$ and $y = h/\sqrt{2}$ have equal areas. After some more (messy) work, we find that

$$Q = \left(\frac{-b^2}{2\alpha}, \frac{hb}{2\alpha}\right).$$

Week 10. *(a) Proposed by Ryan Berndt; (b) 1999 ECC Problem 5.*

(a) Show that the formula

$$\int_{-1}^{1} p(x)\, dx = p(-\sqrt{3}/3) + p(\sqrt{3}/3)$$

yields exact results for polynomials of degree three or less.

(b) (i) Find the points x_1 and x_2 so that the formula

$$\int_0^1 p(x)\,dx = p(x_1) + p'(x_2)$$

yields exact results for polynomials of degree two or less.

(ii) Determine the error in using the resulting formula for a third degree polynomial $p(x)$ with leading coefficient 1.

Solution. For (a), let $p(x) = ax^3 + bx^2 + cx + d$. Then

$$\begin{aligned}\int_{-1}^1 p(x)\,dx &= \left.\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right|_{-1}^1 \\ &= \frac{2b}{3} + 2d \\ &= p(-\sqrt{3}/3) + p(\sqrt{3}/3).\end{aligned}$$

For (b)(i), let $p(x) = ax^2 + bx + c$. Then

$$\begin{aligned}\int_0^1 p(x)\,dx &= \left.\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right|_0^1 \\ &= \frac{a}{3} + \frac{b}{2} + c,\end{aligned}$$

and

$$\begin{aligned}p(x_1) + p'(x_2) &= ax_1^2 + bx_1 + c + 2ax_2 + b \\ &= a(x_1^2 + 2x_2) + b(x_1 + 1) + c.\end{aligned}$$

Equating coefficients of a and b gives us $x_1 = -1/2$ and $x_2 = 1/24$.

For (b)(ii), let $p(x) = x^3 + ax^2 + bx + c$. Leaving off the details, the error we seek is given by

$$\left|\int_0^1 p(x)\,dx - p\left(-\frac{1}{2}\right) - p'\left(\frac{1}{24}\right)\right| = \frac{71}{192}.$$