Coffee Hour Problems of the Week

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Otterbein College

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Week 1. *Proposed by Matthew McMullen*

You and a friend order a perfectly round 14-inch (in diameter) pizza cut into six congruent slices. The 14 inches includes a 1-inch-wide ring of outer crust. There is one piece left that you want to split, but you are tired of the crust and don't want any more of it. Describe all ways to divide the non-crust part of the slice into two equal-area pieces, one of which has no part of the crust on it, using a single straight-line cut.

Solution. Since the entire non-crust part of the pizza has area 36π, each slice has a "non-crust" area of 6π. Therefore, we wish to divide the slice into two pieces, each with non-crust area 3π, and one of which has no part of the crust. For convenience, let's represent the non-crust part of the slice as the region, R, in the plane, bounded by $y = \sqrt{3}x$, $y = -\sqrt{3}x$, and $y = \sqrt{36 - x^2}$.

One way to bisect R is with a line, l, through the point $(3, 3\sqrt{3})$. One of the resulting pieces would then be a triangle with base 6. Since the area of this triangle must be 3π, its height must be π. Therefore, l is the line through the points $(3, 3\sqrt{3})$ and $(-\pi/\sqrt{3}, \pi)$.

More generally, fix $2\pi/\sqrt{3} \leq \alpha \leq 6$ and let P be the point $(\alpha/2, \sqrt{3}\alpha/2)$. We wish to bisect R with a line, l, through P. One of the resulting pieces will be a triangle with base α; and so its height must be $6\pi/\alpha$. Therefore, l is the line through the points P and $(-2\sqrt{3}\pi/\alpha, 6\pi/\alpha)$. (We better get an accurate ruler!)

Week 2. *Proposed by Matthew McMullen*

Find all four-digit numbers, N, such that the number formed by writing the digits of N backwards is a multiple of N.

Solution. Obviously, N can be any four-digit palindrome, but are there any other solutions? Let $\overline{N}$ be the number formed by writing the digits of N backwards. We want to find all solutions to the equation $\overline{N} = kN$, where k is a positive integer. We have already dealt with the case $k = 1$; and if $k \geq 10$, then kN has more digits than $\overline{N}$. Therefore we need only consider $k = 2, \ldots, 9$.

First suppose $k \in \{5, 6, 8\}$. Then N must start with a 1 and $\overline{N}$ must end in a 1. But this is impossible since kN cannot end in a 1.

Next suppose $k = 7$. Then N must start in a 1 and $\overline{N}$ must end in a 1. Thus, N must end in a 3 and $\overline{N}$ must start with a 3, which is impossible since $kN > 7000$.

Similar reasoning shows that k cannot be 2 or 3. The only values of k that cannot be immediately ruled out are 4 and 9. If $k = 9$, then N must start with a 1 and $\overline{N}$ must end in a 1. So, N must end in a 9. Continuing in this manner, we see that $\boxed{N = 1089}$ is the only solution ($9801 = 9 \times 1089$). By similar logic, if $k = 4$, then $\boxed{N = 2178}$ is the only solution ($8712 = 4 \times 2178$).

Week 3. *2010 AIME II.*

Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

Solution. Put $P(x) = x^4 - nx + 63$. P has a linear factor if and only if it has a root. Suppose α is a root of P. Then $\alpha^4 - n\alpha + 63 = 0$, or

$$n = \alpha^3 + \frac{63}{\alpha}.$$

Thus, α must be a positive divisor of 63; and so the smallest value of n in this case is 48 (when $\alpha = 3$).

We must also consider the possibility that P has two quadratic factors, say $P(x) = (x^2 + ax + b)(x^2 + cx + d)$. Multiplying this out and equating coefficients of like powers of x tells us that $b + d = a^2$ and $bd = 63$. So either $\{b, d\} = \{1, 63\}$ or $\{b, d\} = \{7, 9\}$ The later gives us the smallest possible value of n, namely $\boxed{n = 8}$.

Week 4. *Proposed by Matthew McMullen.*

Let $f(x) = \sqrt{x}$ and $g(x) = x + a$, where $a \neq 0$. Suppose there exists an x_0 such that $f(x_0) = g(x_0)$ and $f(f(x_0)) = g(g(x_0))$. Find a.

Solution. Put $u = \sqrt[4]{x_0}$. We are given that $u^2 = u^4 + a$ and $u = u^4 + 2a$. Solving the first equation for a and plugging this result into the second equation yields $u = u^4 + 2(u^2 - u^4)$. Rearranging and factoring tells us that

$$u(u-1)(u^2 + u - 1) = 0.$$

Since $a \neq 0$, $u \neq 0, 1$. Also, since u is a fourth root, $u \geq 0$. Thus,

$$u = \frac{-1+\sqrt{5}}{2};$$

and so $a = u^2(1-u^2) = \boxed{-2+\sqrt{5}}$.

Week 5. *Proposed by Matthew McMullen* (*inspired by* AMM).

You have five balls, numbered 1 to 5, that you will put into five urns, also numbered 1 to 5. First, ball 1 is put in a randomly selected urn. Then, ball 2 is put in urn 2 if it is empty, otherwise it is put in a randomly selected empty urn. Then, ball 3 is put in urn 3 if it is empty, otherwise it is put in a randomly selected empty urn. Then, ball 4 is put in urn 4 if it is empty, otherwise it is put in a randomly selected empty urn. Finally, ball 5 is put in the last empty urn. The random variable X represents the number of balls whose number matches the number of the urn it is put in. Find the expected value of X.

Solution. If ball 1 is put in urn 1 (which happens with probability 1/5), then all of the balls will be put in their proper urns. If ball 1 is put in urn 5 (which happens with probability 1/5), then only balls 2, 3, and 4 will be put in their proper urns.

If ball 1 is put in urn 4, then balls 2 and 3 will be put in their proper urns and ball 4 is put in either urn 1 or 5. So our final ball placements would be 42315 or 52314, both occurring with probability 1/10.

If ball 1 is put in urn 3, then we have four possible outcomes: 32145 and 52143 (both with probability 1/15), and 42135 and 52134 (both with probability 1/30).

If ball 1 is put in urn 2, then we have eight possible outcomes: 21345 and 51342 (probability 1/20), 41325 and 51324 (probability 1/40), 31245 and 51243 (probability 1/60), and 41235 and 51234 (probability 1/120).

Let $p(x) = \text{Prob}(X = x)$. Putting everything together, we see that $p(0) = 1/120$, $p(1) = 1/12$, $p(2) = 7/24$, $p(3) = 5/12$, $p(4) = 0$, and $p(5) = 1/5$. Thus, the expected value of X is $\boxed{35/12}$.

Week 6. *Proposed by Matthew McMullen.*

Find the maximum value of

$$\sum_{n=0}^{\infty} (-1)^{\lfloor n/2 \rfloor} x^n,$$

where $-1 < x < 1$.

Solution. Let $S(x)$ denote the sum in question. Since S converges absolutely for $|x| < 1$,

$$\begin{aligned} S(x) &= (1+x) - (x^2+x^3) + (x^4+x^5) - (x^6+x^7) + \cdots \\ &= (1+x)(1-x^2+x^4-x^6+\cdots) \\ &= \frac{1+x}{1+x^2}. \end{aligned}$$

Therefore,

$$S'(x) = \frac{-(x^2+2x-1)}{(1+x^2)^2};$$

and so the maximum value of S occurs when $x = -1+\sqrt{2}$. Thus, the maximum value of S is $S(-1+\sqrt{2}) = \boxed{(1+\sqrt{2})/2}$.

Week 7. *Proposed by Matthew McMullen* (*inspired by* College Math. J.).

Prove that a differentiable function on the real line is a quadratic function if and only if the intersection of any two of its tangent lines lies midway horizontally between the points of tangency.

Solution. Let f be differentiable on the real line. The above condition is equivalent to the fact that, for any $x_1 \neq x_2$, the solution to

$$f'(x_1)(x-x_1) + f(x_1) = f'(x_2)(x-x_2) + f(x_2)$$

is $x = (x_1+x_2)/2$. In other words, we need to show that f is quadratic if and only if

$$\frac{f(x)-f(y)}{x-y} = \frac{f'(x)+f'(y)}{2}, \tag{1}$$

for all $x \neq y$.

It is easy to check that (1) holds for quadratic functions f. To show the converse, suppose that (1) holds. Fixing y and differentiating with respect to x yields (after some simplification)

$$f''(x) = \frac{f'(x)-f'(y)}{x-y},$$

for all $x \neq y$. Differentiating with respect to x a second time (and simplifying) yields $f'''(x) = 0$, for all $x \neq y$. Since y is arbitrary, we actually have that $f''' \equiv 0$. Therefore, f is quadratic.

($\Rightarrow$ *direction proven by Melissa Tress.*)

Week 8. *From the 2010 Harvard-MIT Mathematics Tournament.*

Calculate

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n\cdot k!}.$$

Solution. We have

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n\cdot k!} &= \sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=2}^{\infty}\frac{1}{k^n} \\
&= \sum_{k=2}^{\infty}\frac{1}{k!}\cdot\frac{1}{k^2}\cdot\frac{1}{1-\frac{1}{k}} \\
&= \sum_{k=2}^{\infty}\frac{1}{k!}\cdot\frac{1}{k(k-1)} \\
&= \sum_{k=2}^{\infty}\frac{1}{k!}\left(\frac{1}{k-1}-\frac{1}{k}\right) \\
&= \sum_{k=2}^{\infty}\frac{1}{k!}\cdot\frac{1}{k-1}-\sum_{k=2}^{\infty}\frac{1}{k!}\cdot\frac{1}{k} \\
&= \sum_{k=1}^{\infty}\frac{1}{(k+1)!}\cdot\frac{1}{k}-\sum_{k=2}^{\infty}\frac{1}{k!}\cdot\frac{1}{k} \\
&= \frac{1}{2}+\sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{1}{(k+1)!}-\frac{1}{k!}\right) \\
&= \frac{1}{2}-\sum_{k=2}^{\infty}\frac{1}{(k+1)!} \\
&= \frac{1}{2}-\left(\sum_{k=0}^{\infty}\frac{1}{k!}-\sum_{k=0}^{2}\frac{1}{k!}\right) \\
&= \frac{1}{2}-e+1+1+\frac{1}{2} \\
&= \boxed{3-e}.
\end{aligned}$$

Week 9. *From the 1993 Putnam Exam.*

The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant, creating two regions: the first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection. Find c so that the areas of these two regions are equal.

Solution. Let a and b denote the x-coordinates of the intersection points (in the first quadrant) of $y = c$ and the curve, where $a < b$. Then we need to find c such that

$$\int_0^a (c - 2x + 3x^3)\, dx = \int_a^b (2x - 3x^3 - c)\, dx.$$

This equation simplifies to $b = 3c/2$, and we already know that $2b - 3b^3 = c$. Putting these two equations together gives us $\boxed{c = 4/9}$.

(*Solved by Denise Wolfe.*)

Week 10. *Proposed by Matthew McMullen.*

Suppose that $y \geq 3$ and

$$\ln\left(\frac{x}{y}\right) = \frac{\ln x}{\ln y}.$$

Find the minimum possible value of x.

Solution. We may write $x = e^a$ and $y = e^b$, where a is any real number and $b \geq \ln 3$. Since $f(u) = e^u$ is an increasing function, our problem can be solved by minimizing

$$a = \frac{b^2}{b-1}$$

on the interval $[\ln 3, \infty)$. Since

$$\frac{da}{db} = \frac{b(b-2)}{(b-1)^2},$$

the minimum value of a occurs when $b = 2$. Thus, the minimum value of a is 4; so the minimum possible value of x is $\boxed{e^4}$.