Coffee Hour Problems of the Week

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Otterbein College

Autumn 2009

Week 1. *Proposed by Matthew McMullen.*

To gain admittance to a clandestine math society, you need to determine the secret number. This number has two digits and is **uniquely** determined by the answers to the following yes/no questions.

1. Is the number divisible by 2?
2. Is the number divisible by 3?
3. Is the number divisible by 5?
4. Is the number divisible by 7?

(a) What are the answers to the last three questions?

(b) The answer to the first question is "no". What is the secret number?

Solution. Since each of the four questions has two possible answers, there are sixteen possible sets of answers. Only two of these yield a unique number: NNYY gives 35 and YNYY gives 70. Therefore, the answers to the last three questions are "no", "yes", and "yes". In (b) we are told that the number is odd, so the secret number is $\boxed{35}$.

Week 2. *Proposed by Adam Wolfe and Matthew McMullen.*

A circle of radius 1 is externally tangent to a circle of radius 3 at the point P. Let l denote a line tangent to both circles that does not pass through P. Find the area of the region bounded by l and the two circles.

Solution. Let T denote the (right) trapezoid determined by l, the line through the centers of both circles, and the two radii that are perpendicular to l. Let β denote the acute angle in T, and let α denote the obtuse angle in T. By the Pythagorean Theorem, the height of T is $2\sqrt{3}$; thus, the area of T is $4\sqrt{3}$.

The area that we seek is the area of T minus the area of the circular segments inside T. Using trig we see that $\beta = 60°$ and $\alpha = 120°$. Thus, the area is

$$4\sqrt{3} - \frac{1}{3} \cdot \pi - \frac{1}{6} \cdot 9\pi = \boxed{4\sqrt{3} - \tfrac{11\pi}{6}}.$$

Solved by Denise Wolfe.

Week 3. *Proposed by Matthew McMullen.*

Which is larger, $2009^{2010^{2009}}$ or $2010^{2009^{2010}}$?

Solution. By taking the log of the log of both numbers, we see that the question is equivalent to finding the larger of

$$2009 \log 2010 + \log \log 2009$$

and

$$2010 \log 2009 + \log \log 2010.$$

Since the function $y = \log \log x$ (for $x > 1$) is increasing, $\log \log 2009 < \log \log 2010$. Also, notice that the function $y = \frac{\log x}{x}$ is decreasing for $x > e$. Thus,

$$\frac{\log 2010}{2010} < \frac{\log 2009}{2009}.$$

Rearranging this gives

$$2009 \log 2010 < 2010 \log 2009.$$

Thus, $\boxed{2010^{2009^{2010}}}$ is the larger number.

Week 4. *Proposed by Matthew McMullen.*

(a) Let (a_n) be a sequence of real numbers such that the sequence $(2a_{n+1} - a_n)$ converges. Does (a_n) necessarily converge?

(b) Let (a_n) be a sequence of real numbers such that the sequence $(a_{n+1} - 2a_n)$ converges. Does (a_n) necessarily converge?

Solution. The answer to $\boxed{\text{(b) is no}}$. Consider the sequence defined by $a_n = 2^n$ for all n. Then (a_n) diverges and $(a_{n+1} - 2a_n)$ is the zero sequence.

The answer to $\boxed{\text{(a) is yes}}$. Suppose $2a_{n+1} - a_n \to l$ as $n \to \infty$. Then, for all $\epsilon > 0$ there exists a positive integer N such that

$$l - \epsilon < 2a_k - a_{k-1} < l + \epsilon$$

for all $k > N$. Combining these inequalities for $k = N+1, N+2, \ldots, n$ gives

$$(l-\epsilon)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-N}}\right)+\frac{a_N}{2^{n-N}} < a_n < (l+\epsilon)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-N}}\right)+\frac{a_N}{2^{n-N}}$$

for all $n > N$. Thus,

$$l-\epsilon \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le l+\epsilon.$$

Since ϵ is arbitrary, we have shown that (a_n) converges to l.

Week 5. *Proposed by Matthew McMullen.*

In solving $ax^2+bx+c=0$, for $a \neq 0$, some students will subtract c from both sides, factor an x out of the left-hand side, set each factor equal to $-c$, and then solve these two equations! Classify all such quadratic equations where this erroneous method yields *both* correct solutions.

Solution. If $c = 0$, then the "erroneous" method is actually a correct way to solve the equation. For the interesting case, assume $c \neq 0$.

The erroneous method yields the "solutions" $x = -c$ and $x = -(b+c)/a$. Assuming these are correct, we plug them back into the original equation and see that we must have $ac - b + 1 = 0$ and $a + b + c = 0$. Some algebraic manipulation shows that $(a+1)(c+1) = 0$; so, either $a = -1$ or $c = -1$ (or both). If $c = -1$, however, the solutions both reduce to $x = 1$, which can only happen if $a = -1$.

We see, therefore, that the quadratic equations we seek are those where $\boxed{\text{either } c = 0}$, or those of the form

$$\boxed{-x^2+bx+1-b=0},$$

where b is any real number.

Week 6. *Proposed by Matthew McMullen.*

The weatherman issues the following statement: "There is a 40% chance of rain on Saturday and a 50% chance of rain on Sunday. If it rains on Saturday, however, it will be twice as likely to rain on Sunday than if it doesn't rain on Saturday." What is the probability that it will rain at least one day this weekend?

Solution. Let A be the event that it rains on Saturday and B be the event that it rains on Sunday. Let $x = P(A \cap B)$. Then $P(A^c \cap B) = 0.5 - x$.

We are told that $P(B|A) = 2 \cdot P(B|A^c)$. Therefore,

$$\frac{x}{0.4} = 2 \cdot \frac{0.5 - x}{0.6}.$$

Solving this for x yields $x = 2/7$. Thus,

$$P(A \cup B) = \frac{2}{5} + \frac{1}{2} - \frac{2}{7} = \boxed{\tfrac{43}{70}}.$$

Week 7. *1987 Putnam problem B-1.*

Find

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\, dx.$$

Solution. Let I denote the integral in question. Using the substitution $u = 9-x$ gives

$$I = \int_5^7 \frac{\sqrt{\ln u}}{\sqrt{\ln u} + \sqrt{\ln(12-u)}}\, du.$$

Using the substitution $u = x + 3$ gives

$$I = \int_5^7 \frac{\sqrt{\ln(12-u)}}{\sqrt{\ln(12-u)} + \sqrt{\ln u}}\, du.$$

Adding these together yields

$$2I = \int_5^7 1\, du = 2.$$

Thus, $\boxed{I = 1}$.

Week 8. *Proposed by Matthew McMullen.*

Find a continuous function $f(x)$ such that

$$2009 = \int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx = \int_0^1 x^2 f(x)\, dx.$$

Solution. We guess that one solution may be of the form $f(x) = ax^2 + bx + c$. Plugging this into the given system of equations yields

$$2009 = \frac{a}{3} + \frac{b}{2} + c = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} = \frac{a}{5} + \frac{b}{4} + \frac{c}{3}.$$

Solving for a, b, and c gives $a = 10(6027)$, $b = -8(6027)$, and $c = 6027$. Thus, $\boxed{f(x) = 6027(10x^2 - 8x + 1)}$ is one possible solution.

Week 9. *2009 Harvard-MIT Mathematics Tournament.*

Let $\mathcal{R}$ be the region in the plane bounded by the graphs of $y = x$ and $y = x^2$. Compute the volume of the region formed by revolving $\mathcal{R}$ around the line $y = x$.

Solution. Let V be the volume in question. For each $x \in [0, \sqrt{2}]$, we can rotate the point $(x, 0)$ forty-five degrees counterclockwise to the point $P_x = (x/\sqrt{2}, x/\sqrt{2})$. Let l_x be the line segment with slope -1 connecting P_x to the graph of $y = x^2$, and let r_x be the length of l_x. Then

$$V = \pi \int_0^{\sqrt{2}} r_x^2 \, dx.$$

After some (messy) algebra, we see that the x-coordinates of the endpoints of l_x are

$$\frac{x}{\sqrt{2}} \quad \text{and} \quad \frac{-1 + \sqrt{1 + 4\sqrt{2}\,x}}{2}.$$

Since the slope of l_x is -1 we have

$$\begin{aligned} r_x^2 &= 2\left(\frac{x}{\sqrt{2}} - \frac{-1 + \sqrt{1 + 4\sqrt{2}\,x}}{2}\right)^2 \\ &= x^2 + 3\sqrt{2}\,x + 1 - (1 + \sqrt{2}\,x)\sqrt{1 + 4\sqrt{2}\,x}. \end{aligned}$$

Using the substitution $u = x/\sqrt{2}$ gives

$$\begin{aligned} V &= \pi \int_0^{\sqrt{2}} r_x^2 \, dx \\ &= \pi\sqrt{2} \int_0^1 (2u^2 + 6u + 1 - (1 + 2u)\sqrt{1 + 8u}) \, du \\ &= \boxed{\tfrac{\pi\sqrt{2}}{60}}. \end{aligned}$$

Week 10. *Proposed by Matthew McMullen.*

The polynomial

$$f(x) = x^4 - 2x^3 - 2009x^2 + \alpha x + \frac{11}{13}$$

is symmetric about some vertical line. Find α.

Solution. Suppose that $f(x)$ is symmetric about the line $x = A$. Then $f(A+x) = f(A-x)$ for all x. Plugging this in and equating coefficients of like powers of x gives

$$8A - 4 = 0 \text{ and } 8A^3 - 12A^2 - 8036A + 2\alpha = 0.$$

Then $A = 1/2$ and $\boxed{\alpha = 2010}$. (See you next year!)

Solved by Sean Poncinie.