Coffee Hour Problems of the Week

Edited by Matthew McMullen

Otterbein College

Winter 2010

Week 1. *Proposed by Matthew McMullen.*

Show that $\sqrt{2009} + \sqrt{2010}$ is a root of a fourth degree polynomial with integer coefficients. Is there a non-zero polynomial with integer coefficients and of degree *less than* four that $\sqrt{2009} + \sqrt{2010}$ is a root of?

Week 2. *Proposed by Matthew McMullen.*

(a) Let $m \geq n \geq 0$. Show that

$$\sum_{i=n}^{m} \binom{i}{n} = \binom{m+1}{n+1}.$$

(b) Find

$$\sum_{i=1}^{1729} \binom{3739-i}{2010} \quad \text{and} \quad \sum_{i=1}^{1729} i\binom{3739-i}{2010}.$$

Week 3. *Proposed by Matthew McMullen.*

If N is a positive integer with at least two prime divisors, define the *delta value* of N to be $p - q$, where $q < p$ are the two largest prime divisors of N. Find the previous five years and the next five years with the same delta value as 2010.

Week 4. *Proposed by Matthew McMullen.*

Let $F_1 = 1 = F_2$ and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 3$. (So F_n is the nth Fibonacci number.) Find all n such that $F_n = n^2$.

Week 5. *Proposed by Matthew McMullen.*

(a) When I type i^i into my TI-83 calculator, it gives me 0.2078795764. When I type in $(-i)^i$, it gives me 4.810477381. What are the exact values of these numbers? More generally, how would you "make sense" of z^w, where z and w are complex numbers (and $z \neq 0$)?

(b) Classify all complex numbers z and w with $|z| = 1$ and $z^w \in \mathbb{R}$.

Week 6. *Proposed by Matthew McMullen.*

Suppose

$$\int_0^a \frac{1}{\sqrt{1+\sqrt{x}}}\, dx = 2010.$$

Find, with minimal computational aid, the first two digits of a.

Week 7. *2009 Ohio MAA Student Team Competition.*

Let P be a point picked at random inside the equilateral triangle ABC. What is the probability that the angle $\angle APB$ is an acute angle?

Week 8. *Proposed by Matthew McMullen.*

Find

$$\sum_{n=1}^{\infty} \frac{2n-1}{(4n-1)!}\,.$$

Week 9. *Proposed by Matthew McMullen.*

Let T be the region bounded by an isosceles triangle. Mathematically describe all ways of dividing T into two equal-area pieces using a straight line.

Week 10. *(a) Proposed by Ryan Berndt; (b) 1999 ECC Problem 5.*

(a) Show that the formula

$$\int_{-1}^{1} p(x)\,dx = p(-\sqrt{3}/3) + p(\sqrt{3}/3)$$

yields exact results for polynomials of degree three or less.

(b) (i) Find the points x_1 and x_2 so that the formula

$$\int_{0}^{1} p(x)\,dx = p(x_1) + p'(x_2)$$

yields exact results for polynomials of degree two or less.

(ii) Determine the error in using the resulting formula for a third degree polynomial $p(x)$ with leading coefficient 1.