Coffee Hour Problems of the Week

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Week 1. *Proposed by Matthew McMullen.*

Show that $2009 = p^2 \cdot q$, where p and q are distinct prime numbers, and find the next five years that will factor in this way.

Solution. If we start listing different values for p and q (having a table of primes is helpful), we see that the next five years that factor as $p^2 \cdot q$ are:

$$\begin{aligned} 2012 &= 2^2 \cdot 503 \\ 2023 &= 17^2 \cdot 7 \\ 2036 &= 2^2 \cdot 509 \\ 2043 &= 3^2 \cdot 227 \\ 2057 &= 11^2 \cdot 17. \end{aligned}$$

Solved by Sean Poncinie, Zach Thomas, and Justin Young. Delisa Mason found three of the years.

Week 2. *Proposed by Matthew McMullen.*

The Erdős-Straus conjecture (an unsolved problem since 1948) states that for all integers $n \geq 2$, there exist positive integers x, y, and z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z};$$

for example,

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}.$$

Show that this conjecture is true for $n = 2009$. (**Bonus:** Find a solution with x, y, and z all distinct.)

Solution. There are many solutions to this problem. One particularly elegant one is:

$$\begin{aligned}
\frac{4}{2009} &= \frac{1}{287} \cdot \frac{4}{7} \\
&= \frac{1}{287} \cdot \frac{24}{42} \\
&= \frac{1}{287} \cdot \frac{14 + 7 + 3}{42} \\
&= \frac{1}{287} \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{14} \right) \\
&= \frac{1}{861} + \frac{1}{1722} + \frac{1}{4018}.
\end{aligned}$$

Solved by Delisa Mason, Sean Poncinie, and Justin Young.

Week 3. *Proposed by Matthew McMullen.*

For integers $n \geq 2$, $n\#$ (read n *primorial*) is defined as the product of all primes less than or equal to n. For example, $10\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. Prove that

$$2 + 2003\#, 3 + 2003\#, 4 + 2003\#, \ldots, 2010 + 2003\#$$

is a list of 2009 consecutive composite numbers. (*Fun fact:* The first number in the above list has 846 digits!)

Solution. The given list of numbers is clearly a sequence of 2009 consecutive integers. Let $n + 2003\#$ be a number on the list and let p be a prime divisor of n. Since 2005, 2007, and 2009 are composite, $p \leq 2003$. Thus p is one of the primes that make up $2003\#$; therefore, p divides $n + 2003\#$. Since our choice of $n + 2003\#$ was arbitrary, we are done.

Solved by Delisa Mason, Sean Poncinie, and Justin Young.

Week 4. *Proposed by Ryan Berndt and Matthew McMullen.*

It can be shown that if $f(x)$ is integrable on $[0,1]$, then $[f(x)]^2$ is integrable on $[0,1]$ and

$$\int_0^1 f(x)\,dx \le \left(\int_0^1 [f(x)]^2\,dx\right)^{1/2}. \tag{1}$$

(a) Show that

$$\int_0^1 \sqrt{x}\cdot e^{x^2}\,dx < \frac{e}{2}.$$

(b) Research the Arithmetic/Quadratic Means Inequality, and use it to prove inequality (1).

Solution. We will prove **(b)** first. By the Arithmetic/Quadratic Means Inequality,

$$\frac{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)}{n} \le \sqrt{\frac{f^2\left(\frac{1}{n}\right)+f^2\left(\frac{2}{n}\right)+\cdots+f^2\left(\frac{n}{n}\right)}{n}},$$

for all positive integers n. Taking the limit as n approaches infinity proves (1).

Now, by (1),

$$\begin{aligned}
\int_0^1 \sqrt{x}\cdot e^{x^2}\,dx &\le \left(\int_0^1 x\,e^{2x^2}\,dx\right)^{1/2} \\
&\overset{(u=2x^2)}{=} \left(\frac{1}{4}\int_0^2 e^u\,du\right)^{1/2} \\
&= \frac{1}{2}\sqrt{e^2-1} \\
&< \frac{e}{2}.
\end{aligned}$$

Solved by Sean Poncinie and Delisa Mason (part a).

Week 5. *Proposed by Matthew McMullen.*

The minute hand of a clock is twice as long as the hour hand. Find a time when the distance between the tips of the hour and minute hands is increasing at the largest rate.

Solution. We will show that the answer can be any time when the minute hand is $60°$ past the hour hand, e.g. 10:00. Let $a > 0$ be the length of the hour hand. Then, by assumption, the length of the minute hand is $2a$. Let $\theta = \theta(t)$ represent the angle that the minute hand is "ahead" of the hour hand, where $0° \leq \theta \leq 180°$[1]. Let $c = c(t)$ denote the distance between the tips of the hands. By the Law of Cosines,

$$c^2 = a^2 + (2a)^2 - 2a(2a)\cos\theta.$$

Differentiating with respect to time and using the Law of Sines gives

$$c' = 2a^2\theta' \frac{\sin\theta}{c} = 2a^2\theta' \frac{\sin\beta}{2a} = a\theta' \sin\beta,$$

where β is the angle opposite the minute hand.

Since a and θ' are constant, the maximum value of c' occurs when $\beta = 90°$; i.e. when

$$\theta = \arccos\left(\frac{a}{2a}\right) = 60°.$$

Solved by Sean Poncinie.

Week 6. *Proposed by Ryan Berndt.*

Your linear algebra teacher gives you two 5×5 matrices, A and B, and asks whether or not they are inverses of each other. After some tedious matrix multiplication, you find that $AB = I$, where I denotes the 5×5 identity matrix. According to the definition of matrix inverses you still need to show that $BA = I$ (remember that matrix multiplication is *not* commutative). You are tired and don't want to do any more matrix multiplication. Can you conclude immediately that $BA = I$?

[1] For $180° < \theta < 360°$, the distance between the tips of the hands is *decreasing.*

Solution. The answer is yes! We first claim that A is invertible. Suppose, on the contrary, that A is not invertible. Then $\det(A) = 0$; so,

$$\det(AB) = \det(A)\det(B) = 0.$$

This is a contradiction since we are given that

$$\det(AB) = \det(I) = 1.$$

Now, since $AB = I$, we see that

$$A^{-1}(AB)A = A^{-1}IA.$$

Thus, $BA = I$, as desired.

Solved by Sean Poncinie.

Week 7. *Proposed by Matthew McMullen.*

A top-secret vault is opened by pressing four buttons, conveniently numbered 1-4, in the correct order (without repeats). The lock doesn't reset if the incorrect code is entered. (For example, pressing button 1 then 2 then 3 then 4 then 1 tries two different combinations: 1 2 3 4 and 2 3 4 1.)

(a) You have no idea what the combination is. Assuming optimal strategy, what is the maximum number of button-pushes that are needed to open the vault?

(b) Another vault with an unknown combination has n buttons. Assuming optimal strategy, what is the maximum number of button-pushes that are needed to open the vault?

Solution. (a) After some trial-and-error, we see that the maximum number of button-pushes is 33. One such sequence of button-pushes is

$$123412314231243121342132413214321.$$

(b) Let $a(n)$ denote the maximum number of button-pushes for a vault with n buttons. We see that

$$a(1) = 1, a(2) = 3, a(3) = 9, a(4) = 33, \text{and } a(5) = 153.$$

We conjecture that

$$a(n) = \sum_{k=1}^{n} k!,$$

but cannot prove this result.[2]

Part (a) solved by Sean Poncinie.

Week 8. *Classic problem proposed by Matthew McMullen.*

You are standing on a ladder that is leaned up against a house. Your "friend" on the ground pulls the base of the ladder away from the house at a constant rate. Describe the curve that your body traces out.

Solution. We claim that your body will trace out part of an ellipse. For simplicity, we will assume that the ladder is represented initially as the line segment connecting the points $(0, 0)$ and $(0, l)$, where $l > 0$ is the length of the ladder, and that your body is the point $(0, h)$, where $0 < h < l$ represents your height up the ladder. Also, since the base of the ladder is pulling away from the house at a constant rate, we may assume, without loss of generality, that the ladder is flat on the ground at time $t = 1$.

Let $(x(t), y(t))$ be your position at time t. Momentarily fix $0 < t < 1$. At this point, the base of the ladder is lt units from the house, and, by the Pythagorean Theorem, the top of the ladder is $l\sqrt{1-t^2}$ units from the ground. Using similar triangles, we see that

$$x = x(t) = t(l - h) \text{ and } y = y(t) = h\sqrt{1-t^2}.$$

Thus,

$$\frac{x^2}{(l-h)^2} + \frac{y^2}{h^2} = 1,$$

which describes an ellipse.

Solved by Sean Poncinie.

[2] Ronnie Pavlov, a postdoctoral fellow at the University of British Columbia, has proven that $a(n)$ is *at most* the conjectured result.

Week 9. *Proposed by Matthew McMullen.*

It can be shown that $\sin 80^\circ = \sin 40^\circ + \sin 20^\circ$. Find all angles θ such that $0^\circ \leq \theta < 360^\circ$ and

$$\sin 4\theta = \sin 2\theta + \sin \theta.$$

Solution. The fact that one solution to the equation is one-third of a "nice" angle suggests that we should probably be on the lookout for triple-angle formulas. Notice that

$$\begin{aligned} \sin 4\theta - \sin 2\theta - \sin \theta &= 2 \sin 2\theta \cos 2\theta - 2 \sin \theta \cos \theta - \sin \theta \\ &= 4 \sin \theta \cos \theta (2 \cos^2 \theta - 1) - 2 \sin \theta \cos \theta - \sin \theta \\ &= \sin \theta (8 \cos^3 \theta - 6 \cos \theta - 1) \\ &= \sin \theta (2 \cos 3\theta - 1). \end{aligned}$$

Thus, the angles we are looking for satisfy either $\sin \theta = 0$ or $\cos 3\theta = 1/2$. Therefore, there are eight possible solutions:

$$\theta = 0^\circ, 20^\circ, 100^\circ, 140^\circ, 180^\circ, 220^\circ, 260^\circ, \text{and } 340^\circ.$$

Partially solved by Sean Poncinie.