Coffee Hour Problems of the Week

Matthew McMullen - Editor

Otterbein College

Spring 2009

Week 1. *Proposed by Matthew McMullen.*

Find all integers x and y such that

$$x^2 + 41y^2 = 2009.$$

How many rationals x and y can you find that satisfy this equation?

Solution. The given equation describes the ellipse

$$\frac{x^2}{2009} + \frac{y^2}{49} = 1.$$

Thus, $-7 \leq y \leq 7$. Testing all integers in the range, we find that $(0, -7)$ and $(0, 7)$ are the only integer points on the curve.

Finding all rational points on the curve is much trickier. In fact, if an ellipse has one rational point it has infinitely many, and there is a way to classify *all* such points. The classical approach (in this case) is to fix a rational t and find the second intersection point between the ellipse and the line connecting the points $(t, 0)$ and $(0, 7)$. Not only will this intersection point have rational coefficients, but, as t ranges through all rationals (including the point at infinity), all rational points on the curve will be accounted for.[1]

After careful work, we see that all rational points on the curve are given by $(0, 7)$ and

$$\left(\frac{4018t}{t^2 + 2009}, \frac{7(t^2 - 2009)}{t^2 + 2009} \right),$$

where t ranges through all rational numbers.

Integer points found by Sean Poncinie.

[1] This requires proof, of course, but the proof is not terribly difficult.

Week 2. *Proposed by Matthew McMullen.*

Let C_1 denote a circle of radius 1 centered at the origin, and let C_2 denote a circle of radius 2 centered at the point $(5, 0)$. How many lines are tangent to both circles simultaneously? Find an equation for each of these lines.

Solution. There are four common tangent lines, two interior tangents and two exterior tangents. The four equations are

$$y = \pm\frac{3}{4}\left(x - \frac{5}{3}\right) \quad \text{and} \quad \pm\frac{\sqrt{6}}{12}(x + 5).$$

The easiest way to find these is to use similar triangles. For example, to find the interior tangent with positive slope, we use similar triangles to see that the x intercept of this line is $(5/3, 0)$. Then we use similar triangles and the Pythagorean Theorem to see that another point on the line is $(3/5, -4/5)$, and the equation of the line is easily calculated. In a similar way, we can find the exterior tangents.

Solved by Sean Poncinie.

Week 3. *Proposed by Matthew McMullen.*

The *Pell numbers* are defined by

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{if } n \geq 2 \end{cases}.$$

(a) Find the first twelve Pell numbers.

(b) Prove that, for all $n \geq 1$, a triangle with sides of length $2P_nP_{n+1}$, $P_{n+1}^2 - P_n^2$, and P_{2n+1} is a right triangle with integer sides whose legs differ by 1. (This actually classifies *all* such right triangles!)

Solution (outline). The first twelve Pell numbers are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \text{ and } 5741.$$

First show, using induction on m, that

$$P_{2n+1} = P_{m+1}P_{2n-m+1} + P_mP_{2n-m},$$

for all $n \geq 0$ and $0 \leq m \leq 2n$. In particular, for $m = n$ we have

$$P_{2n+1} = P_{n+1}^2 + P_n^2.$$

Next show, using induction on n, that

$$P_{n+1}^2 - P_n^2 - 2P_nP_{n+1} = (-1)^n,$$

for all $n \geq 0$. Finally,

$$\begin{aligned}(2P_nP_{n+1})^2 + (P_{n+1}^2 - P_n^2)^2 &= P_{n+1}^4 + 2P_n^2P_{n+1}^2 + P_n^4 \\ &= (P_{n+1}^2 + P_n^2)^2 \\ &= P_{2n+1}^2.\end{aligned}$$

Week 4. *Proposed by Matthew McMullen.*

Let

$$I = \int_0^\infty \frac{1}{\sqrt{x} + x^2}\, dx.$$

(a) Show that I exists.

(b) Evaluate I.

Solution. We have that

$$\begin{aligned}I &= \int_0^1 \frac{1}{\sqrt{x} + x^2}\, dx + \int_1^\infty \frac{1}{\sqrt{x} + x^2}\, dx \\ &< \int_0^1 \frac{1}{\sqrt{x}}\, dx + \int_1^\infty \frac{1}{x^2}\, dx \\ &< \infty.\end{aligned}$$

To evaulate I, we first substitute $u^2 = x$ to get

$$I = 2\int_0^\infty \frac{1}{u^3 + 1}\, du.$$

Using partial fractions, we may rewrite this as $I = (2/3)I_1 + I_2$, where

$$I_1 = \int_0^\infty \left(\frac{1}{u+1} - \frac{u - 1/2}{u^2 - u + 1} \right) du \text{ and } I_2 = \int_0^\infty \frac{1}{u^2 - u + 1}\, du.$$

Now,

$$I_1 = \lim_{M\to\infty} \left[\ln(u+1) - \frac{1}{2}\ln(u^2 - u + 1) \right]_0^M = 0.$$

To evaluate I_2, we complete the square and substitute $v = u - 1/2$. Thus,

$$
\begin{aligned}
I &= I_2 \\
&= \int_{-1/2}^{\infty} \frac{1}{v^2 + (\sqrt{3}/2)^2}\, dv \\
&= \lim_{M\to\infty} \frac{2}{\sqrt{3}} \arctan\left(\frac{2v}{\sqrt{3}}\right)\Big|_{-1/2}^{M} \\
&= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} + \frac{\pi}{6}\right) \\
&= \frac{4\pi}{3\sqrt{3}}.
\end{aligned}
$$

Week 5. *Proposed by Matthew McMullen.*

Let n be a positive integer, and let p and q be such that $0 < p < 1$ and $q = 1-p$. For $k = 0, 1, \ldots, n$ define

$$P(k) = \binom{n}{k} p^k q^{n-k}.$$

Show that

$$
\begin{aligned}
\sum_{k=0}^{n} P(k) &= 1, && (1)\\
\sum_{k=0}^{n} kP(k) &= np, \text{ and} && (2)\\
\sum_{k=0}^{n} (k-np)^2 P(k) &= npq. && (3)
\end{aligned}
$$

(So P describes a probability distribution with mean np and variance npq.)

Solution. Equation (1) follows directly from the binomial theorem since

$$\sum_{k=0}^{n} P(k) = (p+q)^n = 1.$$

Equation (2) holds since

$$\begin{aligned}\sum_{k=0}^{n} kP(k) &= np\sum_{k=1}^{n}\frac{1}{np}kP(k)\\ &= np\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}p^k q^{n-k-1}\\ &= np(p+q)^{n-1}\\ &= np.\end{aligned}$$

To prove (3), we first claim that

$$\sum_{k=0}^{n} k^2 P(k) = npq + (np)^2.$$

Notice that this equation is equivalent to

$$\frac{1}{np}\sum_{k=1}^{n} k^2 P(k) = 1 + p(n-1),$$

and

$$\begin{aligned}\frac{1}{np}\sum_{k=1}^{n} k^2 P(k) &= \sum_{k=0}^{n-1}(k+1)\frac{(n-1)!}{k!(n-k-1)!}p^k q^{n-k-1}\\ &= 1+\sum_{k=0}^{n-1}k\frac{(n-1)!}{k!(n-k-1)!}p^k q^{n-k-1}\\ &= 1+p(n-1)\sum_{k=0}^{n-2}\frac{(n-2)!}{k!(n-k-2)!}p^k q^{n-k-2}\\ &= 1+p(n-1)(p+q)^{n-2}\\ &= 1+p(n-1).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n}(k-np)^2P(k) &= \sum_{k=0}^{n}k^2P(k) - 2np\sum_{k=0}^{n}kP(k) + (np)^2\sum_{k=0}^{n}P(k)\\ &= npq + (np)^2 - 2(np)^2 + (np)^2\\ &= npq.\end{aligned}$$

Solved by Sean Poncinie.

Week 6. *Proposed by Matthew McMullen.*

For n a positive integer let

$$P(n) = \sum_{k=1}^{n} \arctan \frac{1}{\sqrt{k}}.$$

Does $\lim_{n\to\infty} P(n)$ exist? If so, prove it; if not, find constants $a > 0$ and r such that

$$\lim_{n\to\infty} \frac{P(n)}{n^r} = a.$$

Solution. Notice that

$$\begin{aligned}
\lim_{k\to\infty} \frac{\arctan(1/\sqrt{k})}{1/\sqrt{k}} &\overset{(u=1/\sqrt{k})}{=} \lim_{u\to 0^+} \frac{\arctan u}{u} \\
&\overset{\text{(l'H)}}{=} \lim_{u\to 0^+} \frac{1}{u^2+1} \\
&= 1.
\end{aligned}$$

Thus, by the Limit Comparison Test, since $\sum_{k=1}^{\infty} 1/\sqrt{k}$ diverges, $P(n)$ diverges.

For the second part, we claim that $r = 1/2$ and $a = 2$. We arrived at this result by comparing $P(n)$ to an appropriate integral, but we will prove it by using the Stolz-Cesàro Theorem (S-C). We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{P(n)}{\sqrt{n}} &\overset{\text{(S-C)}}{=} \lim_{n\to\infty} \frac{\arctan(1/\sqrt{n+1})}{\sqrt{n+1}-\sqrt{n}} \\
&= \lim_{n\to\infty} (\sqrt{n+1}+\sqrt{n}) \left(\frac{1}{\sqrt{n+1}} + o\left(\frac{1}{\sqrt{n}}\right)\right) \\
&= \lim_{n\to\infty} \left(1 + \sqrt{\frac{n}{n+1}}\right) \\
&= 2.
\end{aligned}$$

Week 7. *Proposed by Alex Frentz.*

(a) Write a precise mathematical definition for what it means for a function f to be symmetric about the point (a, b).

(b) Is every cubic polynomial symmetric about some point? Explain (with proof or counterexample).

Solution. The function $f(x)$ is *symmetric about* (a, b) if $f(x+a) - b$ is odd; i.e., if

$$f(-x+a) - b = -(f(x+a) - b).$$

Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial. We can "complete the cube" to show that every cubic polynomial is symmetric about its inflection point. We may write

$$f(x) = a\left(x + \frac{b}{3a}\right)^3 + B\left(x + \frac{b}{3a}\right) + f\left(-\frac{b}{3a}\right),$$

where $B = (3ac - b^2)/(3a)$.

Solved by Sean Poncinie.

Week 8. *Proposed by Matthew McMullen.*

Classify all integers n such that $2^n - n^2$ is divisible by 7.

Solution. Clearly, $n \geq 0$, since otherwise $2^n - n^2$ wouldn't be an integer. By the division algorithm we may write $n = 21q + r$ for nonnegative integers q and r with $0 \leq r < 21$. Working mod 7, we see that

$$\begin{aligned} 2^n - n^2 &= 2^{21q+r} - (21q+r)^2 \\ &= (2^3)^{7q} \cdot 2^r - (21q+r)^2 \\ &\equiv 2^r - r^2. \end{aligned}$$

Trying all possible values of r (and working mod 7) we see that only
$r = 2, 4, 5, 6, 10$, and 15 solve $2^r - r^2 = 0$. Thus, $2^n - n^2$ is divisible by 7 iff n is positive and

$$n \equiv 2, 4, 5, 6, 10, \text{ or } 15 \pmod{21}.$$

Week 9, Problem A. *Proposed by Dave Stucki.*

Find

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Week 9, Problem B. *Proposed by Sean Poncinie and Adam Wolfe.*

Find

$$\int_0^{\pi/2} x\, e^x \sin x\, dx.$$

Solutions. The answer to Problem A is 2. We will show, more generally, that

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2},$$

for $-1 < x < 1$. To see this put

$$f(x) = \sum_{n=1}^{\infty} n x^n.$$

By the ratio test, $f(x)$ converges for $-1 < x < 1$, and clearly $f(0) = 0$. If we integrate $f(x)/x$ we get a geometric series which sums to $x/(1-x)$. Therefore,

$$\begin{aligned} f(x) &= x\left(\frac{x}{1-x}\right)' \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

To solve Problem B we first note that, due to integration by parts,

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x - \cos x).$$

Next, we use integration by parts again (with $u = x$ and $dv = e^x \sin x\, dx$) to get

$$\begin{aligned} \int x e^x \sin x\, dx &= \frac{1}{2} x e^x (\sin x - \cos x) - \frac{1}{2}\int e^x (\sin x - \cos x)\, dx \\ &= \frac{1}{2} x e^x (\sin x - \cos x) + \frac{1}{2} e^x \cos x. \end{aligned}$$

Therefore,

$$\int_0^{\pi/2} x\, e^x \sin x\, dx = \frac{1}{2}\left(\frac{\pi}{2} e^{\pi/2} - 1\right).$$

Week 10. *Proposed by Matthew McMullen.*

Find

$$\lim_{n\to\infty} 4^n \left(2 - \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\ 2\mathrm{s}} \right).$$

Solution. The key to this problem is to note that

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\ 2\mathrm{s}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right).$$

This can be shown using induction and the half angle formula.

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} 4^n \left(2 - \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\ 2\mathrm{s}} \right) &= \lim_{n\to\infty} 4^n \left(2 - 2\cos\left(\frac{\pi}{2^{n+1}}\right)\right) \\
&\overset{(u=\pi/2^{n+1})}{=} \frac{\pi^2}{4} \lim_{u\to 0} 2\cdot \frac{1-\cos u}{u^2} \\
&= \frac{\pi^2}{4}.
\end{aligned}$$