Coffee Hour Problems of the Week

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Autumn 2008

Week 1. *Classic problem proposed by Matthew McMullen.*

Each letter in the following "alphametic" equation represents a different digit from 0 to 9:

$$\text{SEND} + \text{MORE} = \text{MONEY}$$

Find the *unique* value of each of the letters so that the equation is true.

Solution. Since we are adding two four-digit numbers to get a five-digit number, M=1 and S is 8 or 9. Thus, O=0, and there is no carry from the hundred's place; so S=9. We also know that N is one more than E, so R=8.

Next we can go through the five possible values of N (3, 4, 5, 6, or 7) one by one. Omitting the details, we see that the only possibility is N=6. So E=5, D=7, and Y=2.

(*Solved by Denise Wolfe, Donna Irvin, Sean Poncinie, and Justin Young.*)

Week 2. *Found problem proposed by Alex Frentz.*

You have 3000 gallons of water at point A on the north side of the desert. Your friends at point B on the south side of the desert want the water, but point B is 1000 miles away. You do have a camel, however. The camel can carry up to 1000 gallons of water at a time. The problem is, for every mile the camel walks, it *drinks* one gallon of water. How much water can you get to point B?

Solution. We first load the camel with 1000 gallons of water, travel 200 miles, unload 600 gallons, and return to point A. We do the same with another 1000 gallons, and then we take the last 1000 gallons, travel 200 miles, and unload the remaining 800 gallons. So we have 2000 gallons of water 200 miles from point A.

Next we load the camel with 1000 gallons of water, travel 1000/3 miles, unload 1000/3 gallons, and return to the rest of the water. We then take the remaining 1000 gallons, travel 1000/3 miles, and unload 2000/3 gallons. So we have 1000 gallons of water 1600/3 miles from point A.

Finally, we take the 1000 gallons of water, travel the 1400/3 miles to point B, and unload the 1600/3 gallons to the cheers of our thirsty friends.

(*Solved by Justin Young and Sean Poncinie.*)

Week 3. *Found problem proposed by Matthew McMullen.*

A nine-digit number, N, contains each of the digits from 1 through 9. For $k = 1, 2, \dots 9$, the number formed by the first k digits of N is divisible by k. Find N.

Solution. First notice that $k = 1$ and $k = 9$ does not give you any information since every number is divisible by 1 and the digits $1, 2, \dots, 9$ sum to 45, which is divisible by 9. Also, the fifth digit must be a 5, and every other digit, starting with the second one, must be even.

By the divisibility test for 4, the fourth digit must be a 2 or a 6; and, by the divisibility test for 6, the corresponding sixth digit must be an 8 or a 4. Similarly, we can go through the divisibility tests for the other values of k, excluding $k = 7$, to narrow our list of possible Ns to the following ten numbers: 147258963, 741258963, 189654327, 981654327, 789654321, 987654321, 183654729, 381654729, 189654723, and 981654723.

Checking each of these for $k = 7$ gives us the unique answer: 381654729.

(*Solved by Sean Poncinie.*)

Week 4. *Original problem proposed by Greg Oman.*

Show that for any odd prime p there is a unique positive integer n such that $n(p + n)$ is a perfect square.

Solution. Let p be an odd prime. We wish to find a unique n such that

$$n(p+n) = k^2,$$

for some positive integer k. Multiplying this equation by 4 and completing the square in n yields the equivalent equation

$$(2n+p+2k)(2n+p-2k) = p^2.$$

Since $k > 0$ and p is prime, we must have $2n+p+2k = p^2$ and $2n+p-2k = 1$. Adding these two equations and solving for n gives

$$n = \frac{p^2+1-2p}{4} = \left(\frac{p-1}{2}\right)^2.$$

(*Unsolved*!)

Week 5. *Proposed by Matthew McMullen.*

Down's syndrome is present in roughly 1 in 900 births. Amniocentesis is a medical procedure that can be used to test for Down's syndrome in a fetus. This test is 99.5% accurate. A pregnant woman decides to get an amniocentesis, and her fetus tests positive for Down's syndrome. What is the probability that her baby actually has Down's syndrome?

Solution. Let D denote the event that the fetus has Down's syndrome and $+$ the event that the amniocentesis comes back positive for Down's. We are given that

$$P(D) = \frac{1}{900},\ P(D^c) = \frac{899}{900},\ P(+|D) = \frac{995}{1000}, \text{ and } P(+|D^c) = \frac{5}{1000}.$$

By Bayes' Theorem,

$$P(D|+) = \frac{P(D)P(+|D)}{P(D)P(+|D) + P(D^c)P(+|D^c)}.$$

Thus,

$$P(D|+) = \frac{199}{1098} \approx 18.12\%.$$

(*Solved by Zach Thomas, Justin Young* (almost!), *and Sean Poncinie.*)

Week 6. *Proposed by Matthew McMullen.*

The *hailstone function* is a function from the integers to the integers defined as follows: if the input is even, divide by 2; if the input is odd, multiply by 3 and add 1. A famous unsolved conjecture is that every positive integer will eventually iterate to 1 under this function. For example,

$$13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

(a) What happens if you iterate -17 with the hailstone function?

(b) Of all the positive integers from 1 to 120, which takes longest to iterate to 1? How many iterations are needed for this number? (**Hint:** A computer program will help here!)

Solution. **(a)** -17 loops back to itself after 18 iterations.

(b) 97, with 118 iterations, takes the longest to iterate to 1.

(*Solved by Sean Poncinie, Justin Young* (part (b) only), *and Delisa Mason.*)

Week 7. *Proposed by Matthew McMullen.*

The *Witch of Agnesi* (named for the mathematician Maria Agnesi) is any curve of the form

$$y = \frac{8a^3}{x^2 + 4a^2},$$

where $a > 0$.

(a) Show that the area between the Witch and the x-axis is $4\pi a^2$.

(b) Show that the volume of revolution of the Witch about the x-axis is $4\pi^2 a^3$.

(c) Is the surface area of revolution of the Witch about the x-axis finite? Explain.

Solution. **(a)** We have

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} \frac{8a^3}{x^2+4a^2}\,dx \\
&= 2a\int_{-\infty}^{\infty} \frac{1}{(\frac{x}{2a})^2+1}\,dx \\
&\overset{(u=\frac{x}{2a})}{=} 4a^2\int_{-\infty}^{\infty} \frac{1}{u^2+1}\,du \\
&= 4a^2 \arctan u\Big|_{-\infty}^{\infty} \\
&= 4a^2\left(\frac{\pi}{2}-\frac{-\pi}{2}\right) \\
&= 4\pi a^2.
\end{aligned}$$

(b) We have

$$\begin{aligned}
V &= \pi\int_{-\infty}^{\infty}\left(\frac{8a^3}{x^2+4a^2}\right)^2 dx \\
&= 64\pi a^6\int_{-\infty}^{\infty}\frac{1}{(x^2+4a^2)^2}\,dx.
\end{aligned}$$

If you make the substitution $2a\tan\theta = x$ and use the identity $\cos^2\theta = (\cos(2\theta)+1)/2$, then we see (after much careful work) that

$$\begin{aligned}
V &= 8\pi a^3\int_{-\pi/2}^{\pi/2}\frac{\cos(2\theta)+1}{2}\,d\theta \\
&= 4\pi a^3(\sin(2\theta)/2+\theta)\Big|_{-\pi/2}^{\pi/2} \\
&= 4\pi a^3\left(\frac{\pi}{2}-\frac{-\pi}{2}\right) \\
&= 4\pi^2 a^3.
\end{aligned}$$

(c) Since the Witch behaves like $y = 1/x^2$ for x large, it is enough to show that

$$\int_1^{\infty} y\sqrt{1+(y')^2}\,dx < \infty.$$

For x large enough, the above integrand behaves like $1/x^2$, so the surface area is indeed finite.

(*Solved by Sean Poncinie*(parts (a) and (b).)

Week 8. *Proposed by Matthew McMullen.*

(a) Show that if a triangle has sides of length 5, 7, and 8, then one of its angles measures 60°.

(b) How many non-similar triangles with sides of integer length and an angle of 60° can you find?

Solution. **(a)** Notice that

$$7^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5 \cdot \cos 60^\circ.$$

Thus, by the Law of Cosines, a triangle with sides of length 5, 7, and 8 has a 60° angle between the sides of length 5 and 8.

(b) I can find infinitely many such triangles. First, we will find infinitely many non-similar triangles with *rational* sides and a 60° angle; then we will multiply each side by a common denominator.

Let a and b represent the lengths of the sides that make up the 60° angle and let c represent the length of the remaining side, where a, b, and c are rational. By rescaling we may assume, without loss of generality, that $b = 1$. By the Law of Cosines, then, we must have

$$c^2 = a^2 + 1 - a,$$

or, equivalently,

$$(2c + 2a - 1)(2c - 2a + 1) = 3.$$

To find solutions to this equation, let α and β be two rational numbers such that $\alpha \cdot \beta = 3$, and suppose $2c + 2a - 1 = \alpha$ and $2c - 2a + 1 = \beta$. Then

$$c = \frac{\alpha^2 + 3}{4\alpha} \quad \text{and} \quad a = \frac{(\alpha - 1)(\alpha + 3)}{4\alpha}.$$

Therefore, triangles with side lengths $a = (\alpha - 1)(\alpha + 3)$, $b = 4\alpha$, and $c = \alpha^2 + 3$, where $\alpha \geq 2$ is an integer, will have integer sides and a 60° angle (between the sides of length a and b). It is a nice (i.e. tedious) exercise to show that these triangles are similar only for $\alpha = 2$ and $\alpha = 5$. So, for every $\alpha \geq 3$ we have a non-similar triangle with integer sides and an angle of 60°.

(*Solved by Sean Poncinie* (12 triangles).)

Week 9. *Proposed by Matthew McMullen.*

(a) Let M be an invertible square matrix, and let $\lambda \neq 0$. If $M^2 = \lambda M$, find M.

(b) Let $M = \begin{pmatrix} a & -2008 \\ b & c \end{pmatrix}$, where a, b, and c are integers. Find the smallest positive value of b such that $M^2 = \mathbf{0}$, where $\mathbf{0}$ denotes the 2×2 zero matrix.

(c) Find a matrix M of the form $M = \begin{pmatrix} a & -2008 \\ b & c \end{pmatrix}$, where a, b, and c are *non-zero* integers and $M^2 = 2M$.

Solution. (a) Suppose $M^2 = \lambda M$. Since M is invertible, we may multiply both sides of this equation by M^{-1} to obtain $M = \lambda I$, where I denotes the 2×2 identity matrix. In other words, M is the diagonal matrix with λs on the diagonal and 0s everywhere else.

(b) Since the top left entry of M^2 is $a^2 - 2008b$, we must have that $a^2 - 2008b = 0$. Therefore, we are looking for the smallest (non-zero) value of b such that $2008b$ is a perfect square. Since $2008 = 2^3 \cdot 251$, $b = 2 \cdot 251 = \boxed{502}$.
$\left(\text{If you're curious, one such } M \text{ is given by } \begin{pmatrix} 1004 & -2008 \\ 502 & -1004 \end{pmatrix}.\right)$

(c) Equating the top left entries of the resulting matrices gives us $a^2 - 2008b = 2a$. Solving for a gives

$$a = 1 \pm \sqrt{2008b + 1}.$$

Thus, we are looking for a non-zero value of b such that $2008b + 1$ is a perfect square. Using Mathematica (okay, so we cheated a bit!), we find that $b = 4(2008) + 4 = 8036$ works. So, one solution is

$$M = \begin{pmatrix} 4018 & -2008 \\ 8036 & -4016 \end{pmatrix}.$$