Coffee Hour Problems of the Week

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Week 1. *Proposed by Matthew McMullen.*

Show that $2009 = p^2 \cdot q$, where p and q are distinct prime numbers, and find the next five years that will factor in this way.

Week 2. *Proposed by Matthew McMullen.*

The Erdős-Straus conjecture (an unsolved problem since 1948) states that for all integers $n \geq 2$, there exist positive integers x, y, and z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z};$$

for example,

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}.$$

Show that this conjecture is true for $n = 2009$. (**Bonus:** Find a solution with x, y, and z all distinct.)

Week 3. *Proposed by Matthew McMullen.*

For integers $n \geq 2$, $n\#$ (read n *primorial*) is defined as the product of all primes less than or equal to n. For example, $10\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. Prove that

$$2 + 2003\#, 3 + 2003\#, 4 + 2003\#, \ldots, 2010 + 2003\#$$

is a list of 2009 consecutive composite numbers. (*Fun fact:* The first number in the above list has 846 digits!)

Week 4. *Proposed by Ryan Berndt and Matthew McMullen.*

It can be shown that if $f(x)$ is integrable on $[0,1]$, then $[f(x)]^2$ is integrable on $[0,1]$ and

$$\int_0^1 f(x)\,dx \leq \left(\int_0^1 [f(x)]^2\,dx\right)^{1/2}. \tag{1}$$

(a) Show that

$$\int_0^1 \sqrt{x} \cdot e^{x^2}\,dx < \frac{e}{2}.$$

(b) Research the Arithmetic/Quadratic Means Inequality, and use it to prove inequality (1).

Week 5. *Proposed by Matthew McMullen.*

The minute hand of a clock is twice as long as the hour hand. Find a time when the distance between the tips of the hour and minute hands is increasing at the largest rate.

Week 6. *Proposed by Ryan Berndt.*

Your linear algebra teacher gives you two 5×5 matrices, A and B, and asks whether or not they are inverses of each other. After some tedious matrix multiplication, you find that $AB = I$, where I denotes the 5×5 identity matrix. According to the definition of matrix inverses you still need to show that $BA = I$ (remember that matrix multiplication is *not* commutative). You are tired and don't want to do any more matrix multiplication. Can you conclude immediately that $BA = I$?

Week 7. *Proposed by Matthew McMullen.*

A top-secret vault is opened by pressing four buttons, conveniently numbered 1-4, in the correct order (without repeats). The lock doesn't reset if the incorrect code is entered. (For example, pressing button 1 then 2 then 3 then 4 then 1 tries two different combinations: 1 2 3 4 and 2 3 4 1.)

(a) You have no idea what the combination is. Assuming optimal strategy, what is the maximum number of button-pushes that are needed to open the vault?

(b) Another vault with an unknown combination has n buttons. Assuming optimal strategy, what is the maximum number of button-pushes that are needed to open the vault?

Week 8. *Classic problem proposed by Matthew McMullen.*

You are standing on a ladder that is leaned up against a house. Your "friend" on the ground pulls the base of the ladder away from the house at a constant rate. Describe the curve that your body traces out.

Week 9. *Proposed by Matthew McMullen.*

It can be shown that $\sin 80^\circ = \sin 40^\circ + \sin 20^\circ$. Find all angles θ such that $0^\circ \leq \theta < 360^\circ$ and

$$\sin 4\theta = \sin 2\theta + \sin\theta.$$