Coffee Hour Problems of the Week

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Week 1. *Proposed by Matthew McMullen.*

Find all integers x and y such that

$$x^2 + 41y^2 = 2009.$$

How many rationals x and y can you find that satisfy this equation?

Week 2. *Proposed by Matthew McMullen.*

Let C_1 denote a circle of radius 1 centered at the origin, and let C_2 denote a circle of radius 2 centered at the point $(5, 0)$. How many lines are tangent to both circles simultaneously? Find an equation for each of these lines.

Week 3. *Proposed by Matthew McMullen.*

The *Pell numbers* are defined by

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{if } n \geq 2 \end{cases} .$$

(a) Find the first twelve Pell numbers.

(b) Prove that, for all $n \geq 1$, a triangle with sides of length $2P_nP_{n+1}$, $P_{n+1}^2 - P_n^2$, and P_{2n+1} is a right triangle with integer sides whose legs differ by 1. (This actually classifies *all* such right triangles!)

Week 4. *Proposed by Matthew McMullen.*

Let
$$I = \int_0^\infty \frac{1}{\sqrt{x} + x^2}\, dx.$$

(a) Show that I exists.

(b) Evaluate I.

Week 5. *Proposed by Matthew McMullen.*

Let n be a positive integer, and let p and q be such that $0 < p < 1$ and $q = 1 - p$. For $k = 0, 1, \ldots, n$ define
$$P(k) = \binom{n}{k} p^k q^{n-k}.$$

Show that
$$\begin{aligned} \sum_{k=0}^{n} P(k) &= 1, & (1) \\ \sum_{k=0}^{n} kP(k) &= np, \text{ and} & (2) \\ \sum_{k=0}^{n} (k - np)^2 P(k) &= npq. & (3) \end{aligned}$$

(So P describes a probability distribution with mean np and variance npq.)

Week 6. *Proposed by Matthew McMullen.*

For n a positive integer let
$$P(n) = \sum_{k=1}^{n} \arctan \frac{1}{\sqrt{k}}.$$

Does $\lim_{n\to\infty} P(n)$ exist? If so, prove it; if not, find constants $a > 0$ and r such that
$$\lim_{n\to\infty} \frac{P(n)}{n^r} = a.$$

Week 7. *Proposed by Alex Frentz.*

(a) Write a precise mathematical definition for what it means for a function f to be symmetric about the point (a, b).

(b) Is every cubic polynomial symmetric about some point? Explain (with proof or counterexample).

Week 8. *Proposed by Matthew McMullen.*

Classify all integers n such that $2^n - n^2$ is divisible by 7.

Week 9, Problem A. *Proposed by Dave Stucki.*

Find

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Week 9, Problem B. *Proposed by Sean Poncinie and Adam Wolfe.*

Find

$$\int_0^{\pi/2} x\, e^x \sin x \, dx.$$

Week 10. *Proposed by Matthew McMullen.*

Find

$$\lim_{n\to\infty} 4^n \left(2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{n \text{ 2s}} \right).$$