Coffee Hour Problems of the Week

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Week 1. *Classic problem proposed by Matthew McMullen.*

Each letter in the following "alphametic" equation represents a different digit from 0 to 9:

$$\text{SEND} + \text{MORE} = \text{MONEY}$$

Find the *unique* value of each of the letters so that the equation is true.

Week 2. *Found problem proposed by Alex Frentz.*

You have 3000 gallons of water at point A on the north side of the desert. Your friends at point B on the south side of the desert want the water, but point B is 1000 miles away. You do have a camel, however. The camel can carry up to 1000 gallons of water at a time. The problem is, for every mile the camel walks, it *drinks* one gallon of water. How much water can you get to point B?

Week 3. *Found problem proposed by Matthew McMullen.*

A nine-digit number, N, contains each of the digits from 1 through 9. For $k = 1, 2, \ldots 9$, the number formed by the first k digits of N is divisible by k. Find N.

Week 4. *Original problem proposed by Greg Oman.*

Show that for any odd prime p there is a unique positive integer n such that $n(p+n)$ is a perfect square.

Week 5. *Proposed by Matthew McMullen.*

Down's syndrome is present in roughly 1 in 900 births. Amniocentesis is a medical procedure that can be used to test for Down's syndrome in a fetus. This test is 99.5% accurate. A pregnant woman decides to get an amniocentesis, and her fetus tests positive for Down's syndrome. What is the probability that her baby actually has Down's syndrome?

Week 6. *Proposed by Matthew McMullen.*

The *hailstone function* is a function from the integers to the integers defined as follows: if the input is even, divide by 2; if the input is odd, multiply by 3 and add 1. A famous unsolved conjecture is that every positive integer will eventually iterate to 1 under this function. For example,

$$13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

(a) What happens if you iterate -17 with the hailstone function?

(b) Of all the positive integers from 1 to 120, which takes longest to iterate to 1? How many iterations are needed for this number? (**Hint:** A computer program will help here!)

Week 7. *Proposed by Matthew McMullen.*

The *Witch of Agnesi* (named for the mathematician Maria Agnesi) is any curve of the form

$$y = \frac{8a^3}{x^2 + 4a^2},$$

where $a > 0$.

(a) Show that the area between the Witch and the x-axis is $4\pi a^2$.

(b) Show that the volume of revolution of the Witch about the x-axis is $4\pi^2 a^3$.

(c) Is the surface area of revolution of the Witch about the x-axis finite? Explain.

Week 8. *Proposed by Matthew McMullen.*

(a) Show that if a triangle has sides of length 5, 7, and 8, then one of its angles measures 60°.

(b) How many non-similar triangles with sides of integer length and an angle of 60° can you find?

Week 9. *Proposed by Matthew McMullen.*

(a) Let M be an invertible square matrix, and let $\lambda \neq 0$. If $M^2 = \lambda M$, find M.

(b) Let $M = \begin{pmatrix} a & -2008 \\ b & c \end{pmatrix}$, where a, b, and c are integers. Find the smallest positive value of b such that $M^2 = \mathbf{0}$, where $\mathbf{0}$ denotes the 2×2 zero matrix.

(c) Find a matrix M of the form $M = \begin{pmatrix} a & -2008 \\ b & c \end{pmatrix}$, where a, b, and c are *non-zero* integers and $M^2 = 2M$.