Coffee Hour Problems of the Week

(edited by Matthew McMullen)

Otterbein College

Winter 2008

Week 2. *Proposed by Matthew McMullen.*

James, Shawn, and Larsa enter their 10 meter by 10 meter classroom to take a topology test. Their professor, Dr. James, tells them to sit so that the minimum of the distances between any two of them is maximized; i.e., to spread out as much as possible. Explain where they should sit in the room to meet Dr. James' request, and find the minimum of the distances between any two of the students when they are in this configuration.

Solution: First notice that, given any configuration, moving the person closest to the west wall against the west wall will only increase the distance to be maximized. Likewise for the person closest to the east, north, and south walls. After moving the students thusly, we have two cases.

Case 1: Two people are seated in opposite corners. Then the best we can do is put the third person in a vacant corner, and the maximum of the minimum distances is 10 meters.

Case 2: One person is seated in a corner and the other two are against the walls not incident to this corner (one person on each such wall). In this case, it is clear that to meet Dr. James' request, the students must form an equilateral triangle. This means that one student is in the corner, and the other two students are against the walls opposite this corner and 15 degrees in from the incident walls.

In Case 2, we see that the distance between any two of our students is

$$\frac{10}{\cos(\pi/12)} = \frac{20}{\sqrt{2+\sqrt{3}}} \approx 10.353 \text{ meters},$$

which is considerably better than Case 1 and is the solution to our problem.

(*Solved by Sean Poncinie and James Orr.*)

Week 3. *Proposed by Zengxiang Tong.*

Determine all polynomials $P(x)$ such that, for all x,

$$(x-1)P(x+1) - (x+2)P(x) = 0. \tag{1}$$

Solution: Plugging in $x = 1$, we see that $-3P(1) = 0$; thus $P(1) = 0$. Plugging in $x = -2$, we see that $-3P(-1) = 0$; thus $P(-1) = 0$. Plugging in $x = 0$, we see that $-P(1) - 2P(0) = 0$; thus $P(0) = 0$. Therefore, by the zero-factor theorem,

$$P(x) = x(x-1)(x+1)g(x), \tag{2}$$

for some polynomial $g(x)$. We will show that $g(x)$ must be a constant.

Suppose $g(x)$ is not constant. Then

$$\lim_{x\to\infty} g(x) = \pm\infty. \tag{3}$$

When we plug (2) into (1), we see that

$$x(x-1)(x+1)(x+2)(g(x+1) - g(x)) \equiv 0.$$

In particular, this means that $g(x+1) = g(x)$ for all $x > 1$. This contradicts (3). Thus, $g(x)$ is constant; and therefore,

$$P(x) = Ax(x-1)(x+1),$$

where A is any constant.

(*Solved by James Orr and Dave Stucki.*)

Week 4. *Proposed by Matthew McMullen (inspired by a talk at the 2007 Ohio-MAA Fall Meeting).*

Toss a fair coin until you get two heads in a row. Let $P(n)$ denote the probability that you have tossed the coin n times ($n \geq 2$). Find a formula for $P(n)$, and justify your result.

Solution: Call a finite sequence of coin flips *good* if it ends in two heads and has no other occurence of two consecutive heads. For $n \geq 2$, let $G(n)$ denote the number of good sequences of length n. It is easy to check that $G(2) = 1$, $G(3) = 1$, $G(4) = 2$, $G(5) = 3$, $G(6) = 5$, and $G(7) = 8$. (For example, the five good sequences of length 6 are TTHTHH, HTHTHH, THTTHH, TTTTHH, HTTTHH.) This suggests that $G(n) = F_{n-1}$, where F_k denotes the kth Fibonacci number.

To prove this, we will use strong induction. As seen above, our formula works for the first 6 possible values of n. Suppose $G(k-1) = F_{k-2}$ and $G(k) = F_{k-1}$, for $k \geq 6$. Clearly, any good sequence of length $k+1$ must end in either TTHH or HTHH. If it ends in TTHH, the initial $k-3$ flips cannot contain two consecutive heads; we have $G(k)$ of these (all possible initial $k-3$ flips of good sequences of length k). If it ends in HTHH, the initial $k-3$ flips cannot contain two consecutive heads *and* cannot end in H; we have $G(k-1)$ of these (all possible initial $k-3$ flips of good sequences of length $k-1$). Therefore, by the inductive hypothesis, $G(k+1) = F_k$.

We have shown, by induction, that $G(n) = F_{n-1}$; and therefore,

$$P(n) = \frac{F_{n-1}}{2^n}.$$

(*Solved by Sean Poncinie and Shawn Winigman.*)

Week 5. *Proposed by Zengxiang Tong.*

Suppose $x_1, x_2, \ldots, x_7$ are integers such that

$$\sum_{n=1}^{7} x_n^3 = 0.$$

Prove that 3 divides the product $x_1 x_2 \cdots x_7$.

Solution: Suppose that 3 does not divide $x_1 x_2 \cdots x_7$. Then each x_i may be written in the form $3k_i \pm 1$, for some integer k_i. Thus,

$$0 = \sum_{n=1}^{7} x_n^3$$

$$= \sum_{n=1}^{7} (3k_i \pm 1)^3$$
$$= \sum_{n=1}^{7} (27k_i^3 \pm 27k_i^2 + 9k_i \pm 1)$$

Reducing this equation modulo 9 tells us that

$$\sum_{n=1}^{7} \pm 1 \equiv 0 \pmod 9.$$

This is impossible since the maximum value of the sum is 7, the minimum value is -7, and the sum can never be 0 since we have an odd number of ± 1's. Therefore 3 must divide $x_1 x_2 \cdots x_7$.

(*Unsolved*!)

Week 6. *Proposed by Shawn Winigman (from Nick's Mathematical Puzzles).*

Describe the largest semicircle that can be inscribed in the unit square, and find the area of this semicircle.

Solution: The largest semicircle is the inscribed semicircle whose base is rotated 45 degrees from the base of the square. If we let r denote the radius of this semicircle, we see that $r + r/\sqrt{2} = 1$; or, equivalently, $r = 2 - \sqrt{2}$. Therefore, the area of this semicircle is

$$\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2 - \sqrt{2})^2 = \pi(3 - 2\sqrt{2}) \approx 0.539.$$

(*Solved by Sean Poncinie and James Orr.*)

Week 7. *Proposed by Matthew McMullen.*

It is well known that 22/7 is an approximation for π. Find the best rational approximation for π with numerator and denominator both less than 1000 (and positive). (*NB*: Computer or calculator assistance is encouraged.)

Solution: Let a and b denote the numerator and denominator, respectively, of our approximation. After programming a TI-83 to compare all fractions with $314/100 \leq a/b \leq 315/100$ to π (which takes about 54 seconds), we see that

$$355/113 \approx 3.14159292$$

is the best approximation.

(*Solved by Justin Kuss, Shawn Winigman, and James Orr.*)

Week 8. *Proposed by Matthew McMullen.*

Let P be an arbitrary point inside the region bounded by parallelogram $ABCD$. Let a denote the length of line segment $\overline{AP}$, b the length of $\overline{BP}$, c the length of $\overline{CP}$, and d the length of $\overline{DP}$. Prove that $a^2+c^2=b^2+d^2$ if and only if $ABCD$ is a rectangle.

Solution: Suppose, without lose of generality, that $\angle A=\theta\leq\pi/2$. Let y be the length of $\overline{AB}$ and x the length of $\overline{AD}$. First suppose that P is in the region bounded by the rectangle with diagonal $\overline{BD}$. Let l be the line through P and perpendicular to $\overline{BC}$. Then l divides $\overline{BC}$ into two line segments, say of length x_1 and x_2. Similarly, l divides $\overline{AD}$ into two line segments, say of length y_1 and y_2. Also, let h_1 be the distance from P to $\overline{BC}$, and let h_2 be the distance from P to $\overline{AD}$.

Then, by the Pythagorean Theorem and some algebra and trigonometry,

$$\begin{aligned}\left(a^2+c^2\right)-\left(b^2+d^2\right) &= \left(y_1^2+h_2^2+x_2^2+h_1^2\right)-\left(x_1^2+h_1^2+y_2^2+h_2^2\right)\\ &= y_1^2-y_2^2+x_2^2-x_1^2\\ &= y_1^2-(x-y_1)^2+(x-x_1)^2-x_1^2\\ &= 2x\left(y_1-x_1\right)\\ &= 2xy\cos\theta.\end{aligned}$$

If P is outside the region bounded by the rectangle with diagonal $\overline{BD}$, then we use a similar argument to get the exact same result.

Therefore, $a^2 + c^2 = b^2 + d^2$ if and only if $2xy \cos \theta = 0$. But the only way this can happen is if $\theta = \pi/2$; i.e. if $ABCD$ is a rectangle.

(*Solved by Sean Poncinie* (*one direction*) *and James Orr.*)

Week 9. *Proposed by Dave Stucki.*

For both of the following sequences, find the missing term and describe the general pattern.

$$15, 23, 27, 29, 30, 39, 43, 45, 46, 51, 53, 54, ?, 58, 60, 71, 75, \ldots$$

$$31, 47, 55, 59, 61, 62, 79, 87, 91, 93, 94, 103, ?, 109, 110, 115, 117, \ldots$$

Solution: The key is to write the sequences of numbers in base two. The first sequence lists (in increasing order) all of the positive integers representable in base two using exactly four ones, and the second sequence lists (in increasing order) all of the positive integers representable in base two using exactly five ones.

Since $54 = 110110_2$, the missing term in the first sequence is $111001_2 = 57$; and, since $103 = 1100111_2$, the missing term in the second sequence is $1101011_2 = 107$.

Can you find a general formula for the nth term in either sequence? Can you find an algorithm that computes the $(n+1)$st term given the nth term?

(*Solved by James Orr and Shawn Winigman.*)