Coffee Hour Problems of the Week

(edited by Matthew McMullen)

Otterbein College

Spring 2008

Week 1. *Proposed by Matthew McMullen.*

Can you write 2008 in the form $a^2 - b^2$, where a and b are positive integers? If so, find *all* such a and b; if not, explain why not.

Solution. We wish to find positive integers a and b such that

$$2008 = a^2 - b^2 = (a+b)(a-b).$$

Since the divisors of 2008 are 1, 2, 4, 8, 251, 502, 1004, and 2008, we have four distinct possibilities:

$$\begin{array}{l} a+b=2008 \\ a-b=1 \end{array}, \quad \begin{array}{l} a+b=1004 \\ a-b=2 \end{array}, \quad \begin{array}{l} a+b=502 \\ a-b=4 \end{array}, \text{ or } \begin{array}{l} a+b=251 \\ a-b=8 \end{array}.$$

Solving the first and last of the above systems of equations yields non-integer solutions, so these can be eliminated. The second system gives $a = 503$ and $b = 501$, while the third system gives $a = 253$ and $b = 249$. Thus

$$2008 = 503^2 - 501^2 = 253^2 - 249^2.$$

(*Solved by James Orr.*)

Week 2. *From the 2008 Iowa Mathematics Competition.*

Write 2008 as a sum of consecutive positive integers.

Solution. We wish to find positive integers x and k such that

$$\begin{aligned} 2008 &= x + (x+1) + \cdots + (x+k) \\ &= (k+1)x + \frac{k(k+1)}{2}. \end{aligned}$$

In other words, we need to find positive integers x and k with

$$x = \frac{2008}{k+1} - \frac{k}{2}.$$

If k is even, then $k+1$ is an odd divisor of $2008 = 2^3 \cdot 251$; i.e. $k = 250$. But $k = 250$ gives a negative value for x. Thus, k must be odd, $k+1$ cannot divide 2008, and $k+1$ must divide $2 \cdot 2008$. Therefore, $k = 15$, which gives $x = 118$; and so

$$2008 = 118 + 119 + \cdots + 133.$$

(*Solved by Sean Poncinie and James Orr.*)

Week 3. *Proposed by Tom James.* (*Quoted from* http://members.cox.net/fathauerart/FractalCrystalArt.html.)

A fractal "is constructed by starting with a [unit] cube and placing a half-scale cube on the center of each face. The second-generation cubes have the same orientation as the first-generation cube. Third-generation cubes again scaled by half are placed on each unoccupied face of a second-generation cube. This process is continued *ad infinitum* to form a 'fractal crystal'."

Find the volume of the fractal crystal.

Solution. First, let c_n denote the number of cubes in each generation. We start with 1 cube with six unoccupied faces, so $c_1 = 1$ and $c_2 = 6$. By the construction of our fractal, each successive generation has five times the number of cubes in the previous generation (one for each of the unoccupied faces). Thus,

$$c_n = 6 \cdot 5^{n-2}$$

for $n \geq 2$.

Next, let v_n denote the volume of each generation-n cube. We start with v_1=1; and since each next-generation cube is scaled by half, the volume gets multiplied by 1/8 from one generation to the next. Thus,

$$v_n = \left(\frac{1}{8}\right)^{n-1}$$

for all positive integers n.

Thus, the total volume of the fractal crystal is given by

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n v_n &= 1 + \sum_{n=2}^{\infty} 6 \cdot 5^{n-2} \cdot \left(\frac{1}{8}\right)^{n-1} \\
&= 1 + \frac{6}{8} \sum_{n=2}^{\infty} 5^{n-2} \cdot \left(\frac{1}{8}\right)^{n-2} \\
&= 1 + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^{n} \\
&= 1 + \frac{3}{4} \cdot \frac{1}{1-(5/8)} \\
&= 3.
\end{aligned}$$

(*Solved by James Orr.*)

Week 4. *Proposed by Matthew McMullen.*

Show that

$$\int_0^1 \Big[\ln(1/x)\Big]^n dx = n!$$

for every nonnegative integer n.

Solution. Since $\ln(1/x) = -\ln x$ for $x > 0$, our problem is equivalent to showing that

$$\int_0^1 (\ln x)^n \, dx = (-1)^n \cdot n! \tag{1}$$

for every nonnegative integer n.

We will proceed by induction. Obviously, (1) is true for $n = 0$. Suppose (1) holds for $n = k \geq 0$. Using integration by parts with $u = (\ln x)^{k+1}$ and

$dv = dx$, we see that

$$\begin{aligned}
\int_0^1 (\ln x)^{k+1}\, dx &= \lim_{\epsilon\to 0^+} x(\ln x)^{k+1}\Big|_\epsilon^1 - (k+1)\int_0^1 (\ln x)^k\, dx \\
&= -\lim_{\epsilon\to 0^+} \frac{(\ln \epsilon)^{k+1}}{1/\epsilon} - (k+1)(-1)^k \cdot k! \\
&= (-1)^{k+1}\cdot (k+1)!\,.
\end{aligned}$$

(The above limit can be seen to be zero by applying L'Hospital's rule and using induction. Details are left to the reader.)

Therefore, by induction, (1) is true for every nonnegative integer n, and we are done.

(*Solved by James Orr.*)

Week 5(a). *Proposed by Matthew McMullen.*

Find all real numbers x such that

$$\arccos x = \arctan x. \tag{2}$$

Solution. First notice that (2) only makes sense for $-1 \le x \le 1$. Also, when $-1 \le x \le 0$, $\arccos x > 0$ but $\arctan x \le 0$; and, when $x = 1$, $\arccos x = 0 \ne \pi/4 = \arctan 1$. Thus, without loss of generality, $0 < x < 1$. Applying the tangent function to both sides of (2) yields the equation

$$\tan(\arccos x) = x. \tag{3}$$

Now, using basic trigonometry, the left-hand side of (3) can be written as $\sqrt{1-x^2}/x$ (this is a good exercise for you). Therefore, we need to solve the equation

$$\frac{\sqrt{1-x^2}}{x} = x,$$

which is equivalent (over the reals) to the equation

$$x^4 + x^2 - 1 = 0.$$

Remembering that $x > 0$, we find that the only solution is

$$x = \sqrt{\frac{-1+\sqrt{5}}{2}}.$$

(*Solved by Sean Poncinie.*)

Week 5(b). *Proposed by Dave Deever.*

Prove that

$$\arctan 1 + \arctan 2 + \arctan 3 = \pi.$$

Solution. Let $a = \arctan 1$, $b = \arctan 2$, and $c = \arctan 3$. Then $\tan a = 1$, $\tan b = 2$, and $\tan c = 3$. By the sum angle formula for tangent, we have

$$\begin{aligned}
\tan(a+b+c) &= \frac{\tan(a+b)+\tan c}{1-\tan(a+b)\tan c} \\
&= \frac{\frac{\tan a+\tan b}{1-\tan a\tan b}+\tan c}{1-\frac{\tan a+\tan b}{1-\tan a\tan b}\tan c} \\
&= \frac{-3+3}{1+9} \\
&= 0.
\end{aligned}$$

Thus, $a+b+c$ is an integer multiple of π. Since $0 < a, b, c < \pi/2$, we know that $0 < a+b+c < 3\pi/2$. Therefore, $a+b+c = \pi$.

(*Unsolved*!)

Week 6. *From the Iowa Mathematics Competition.*

Chicken McNuggets from McDonald's can be ordered in buckets of 6, 9, or 20. If you need 2008 McNuggets, for example, you could order 98 buckets of 20, 4 buckets of 9, and 2 buckets of 6. Find, with proof, the largest number of McNuggets you *couldn't* order.

Solution. We claim that the answer is 43. We need to show that 43 McNuggets cannot be ordered and that any amount larger than 43 *can* be ordered. Suppose that 43 McNuggets could be ordered. Then, since 43 is not a multiple of 3, we need at least one box of 20. But $23 = 43 - 20$ isn't a multiple of 3 either, so we need 2 boxes of 20. But $3 = 43 - 40$ is impossible to achieve. Thus, 43 McNuggets cannot be ordered.

Now suppose $n > 43$. We will consider three cases.

Case 1: n is a multiple of 3. If n is an even multiple of 3, then it can be achieved using just boxes of 6. If n is an odd multiple of 3, then $n - 9$ is an even multiple of 3; thus, n can be achieved using one box of 9 and (zero or more) boxes of 6. (Note that this method will actually work for $n \geq 6$.)

Case 2: n is two more than a multiple of 3. Then $n - 20$ is a multiple of three; so we can use one box of 20 and then apply **Case 1** to $n - 20$. (Note that this method will work for $n \geq 26$.)

Case 3: n is one more than a multiple of 3. Then $n - 40$ is a multiple of three; so we can use two boxes of 20 and then apply **Case 1** to $n - 40$. (Note that this method will work for $n \geq 46$.)

So any amount of McNuggets larger than 43 can be ordered.

(*Solved by James Orr.*)

Week 7. *Proposed by Matthew McMullen.*

A not-so-bright student was asked to solve the following equation (for x) on a MATH 115 exam:

$$\log x - \log 2008 = \log A - \log(x - 516).$$

To solve this equation, he canceled every "log" and solved the resulting equation for x. Amazingly, he got the correct answer! Find A.

Solution. Our intrepid student will solve the equation $x - 2008 = A - (x - 516)$ and obtain

$$x = \frac{2008 + 516 + A}{2}.$$

Obviously, the correct approach is to rewrite the equation as

$$\log \frac{x}{2008} = \log \frac{A}{x-516},$$

and to use the fact that the function $y = \log x$ is one-to-one to obtain

$$\frac{x}{2008} = \frac{A}{x-516}.$$

Cross multiplying and solving for $x > 0$ yields

$$x = \frac{516 + \sqrt{516^2 + 4 \cdot 2008A}}{2}.$$

Equating our solution with our student's solution yields

$$2008 + A = \sqrt{516^2 + 4 \cdot 2008A},$$

which has two solutions: $A = 2524$ or $A = 1492$.

(*Solved by Denise Wolfe and James Orr.*)

Week 8. *Proposed by Matthew McMullen.*

The equation $\sin x = Ax$, where $A > 0$, has exactly nine solutions. Approximate A to five decimal places.

Solution. By examining the graphs of $y = \sin x$ and $y = Ax$, we can see that A is the unique positive integer such that $y = Ax$ is tangent to $y = \sin x$ on the interval $(4\pi, 9\pi/2)$. Now, the line tangent to $y = \sin x$ at $(x_0, \sin x_0)$ is given by

$$y = \cos x_0(x - x_0) + \sin x_0 = (\cos x_0)x + \sin x_0 - x_0 \cos x_0.$$

Thus, we must have $A = \cos x_0$ and $\sin x_0 = x_0 \cos x_0$; i.e. $x_0 = \tan x_0$, where $x_0 \in (4\pi, 9\pi/2)$. Using a graphing calculator, we find that

$$x_0 \approx 14.06619 \text{ and } A \approx 0.07091.$$

(*Solved by Denise Wolfe.*)

Week 9. *Proposed by Matthew McMullen.*

If $f(x)$ and $g(x)$ are differentiable at a and $f(a) = g(a)$, then the **angle between f and g at $x = a$** is defined to be the acute, or possibly right, angle between their tangent lines at $x = a$.

Find, to the nearest tenth of a degree, the angle between $y = x^2$ and $y = \sqrt{x}$ at $x = 1$.

Solution. We wish to find the acute angle, say θ, formed by the lines $l_1 : y = 2x - 1$ and $l_2 y = x/2 + 1/2$. Analyzing the triangles formed by l_1, l_2, $y = 1$, and $x = 2$, we see that

$$\theta = \arctan(2) - \arctan(1/2) \approx 36.870°.$$

(*Solved by Denise Wolfe.*)