Solutions to Coffee Hour Problems

(prepared by Matthew McMullen)

Otterbein College

Autumn 2007

Week 2. *Proposed by Matthew McMullen.*

Describe the nth term in the sequence

$$1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, \ldots$$

and find the next four terms.

Solution: The nth term in the sequence is the smallest positive integer divisible by $1, 2, \ldots, n$. Since 11 and 10 are relatively prime, the eleventh term is $2520(11) = 27720$. Since 27720 is divisible by 12, the twelfth term is also 27720. Since 12 and 13 are relatively prime, the thirteenth term is $27720(13) = 360360$; and since 14 divides this number, this is also the fourteenth term. (Bonus factoid: 232792560 is the smallest whole number divisible by the first twenty positive integers!)

(*Solved by James Orr and Sean Poncinie.*)

Week 3. *Classical problem proposed by Dave Stucki.*

The case of the missing square. See [**1**] for details.

Solution: Magic. Just kidding! See [**1**]. (*Solved by Sean Poncinie.*)

Week 4. *1966 International Mathematics Olympiad problem proposed by Zengxiang Tong.*

Prove that, for all positive integers n, and all valid x,

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

Solution: We will first show

$$\frac{1}{\sin(2^i x)} = \cot(2^{i-1}x) - \cot(2^i x), \tag{1}$$

for all positive integers i and all valid x. To see this, let i be any positive integer. By multiplying through by $\sin(2^i x)$, (1) is equivalent to

$$1 = \sin(2^i x) \cdot \frac{\cos(2^{i-1}x)}{\sin(2^{i-1}x)} - \cos(2^i x). \tag{2}$$

By the double angle formulas (and some algebra), we have

$$\begin{aligned}
\text{R.H.S.} &= 2\sin(2^{i-1}x)\cos(2^{i-1}x) \cdot \frac{\cos(2^{i-1}x)}{\sin(2^{i-1}x)} - (2\cos^2(2^{i-1}x) - 1) \\
&= 2\cos^2(2^{i-1}x) - 2\cos^2(2^{i-1}x) + 1 \\
&= 1.
\end{aligned}$$

This shows (2); and hence, we have proven (1).

Now we are ready to solve our problem by induction. When $n = 1$ our problem is equivalent to (1) when $i = 1$. For the inductive step, we assume

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^k x} = \cot x - \cot 2^k x, \tag{3}$$

for some positive integer k, to prove

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^k x} + \frac{1}{\sin 2^{k+1} x} = \cot x - \cot 2^{k+1} x. \tag{4}$$

Using the inductive hypothesis, (3), this is equivalent to

$$\cot x - \cot 2^k x + \frac{1}{\sin 2^{k+1} x} = \cot x - \cot 2^{k+1} x.$$

By rearranging this is equivalent to

$$\frac{1}{\sin(2^{k+1}x)} = \cot(2^k x) - \cot(2^{k+1}x),$$

which is (1) when $i = k + 1$.

This proves (4); and, by induction, our problem is solved.

(*Solved by James Orr and Adam Wolfe.*)

Week 5. *Proposed by Zengxiang Tong.*

Find

$$\sum_{n=1}^{2007} \frac{5^{2008}}{25^n + 5^{2008}}.$$

Solution: By dividing the numerator and denominator by $5^{2008} = 25^{1004}$ and changing the indices, we see that

$$\begin{aligned} \sum_{n=1}^{2007} \frac{5^{2008}}{25^n + 5^{2008}} &= \sum_{n=1}^{2007} \frac{1}{25^{n-1004} + 1} \\ &= \sum_{n=-1003}^{1003} \frac{1}{25^n + 1}. \end{aligned} \tag{5}$$

Next, notice that, for positive integers k,

$$\begin{aligned} \frac{1}{25^{-k} + 1} + \frac{1}{25^k + 1} &= \frac{25^k}{1 + 25^k} + \frac{1}{25^k + 1} \\ &= \frac{25^k + 1}{25^k + 1} \\ &= 1. \end{aligned}$$

Therefore, we may pair off the first term of summation (5) with the last term, the second term with the next to last term, the third term with the third to last term, etc., to see that

$$\begin{aligned} \sum_{n=-1003}^{1003} \frac{1}{25^n + 1} &= \overbrace{1 + 1 + \cdots + 1}^{1003} + \frac{1}{25^0 + 1} \\ &= \frac{2007}{2}. \end{aligned}$$

(*Solved by Matthew McMullen.*)

Week 6. *Classical problem proposed by Tom James.*

Show that if there are six people in a room, then there are three people that either mutually know each other or mutually are strangers to each other.

Solution: This is a classic result in graph theory; or, more specifically, Ramsey Theory. For positive integers n and m, let $R(n, m)$ denote the minimum number of people that are needed to ensure that either n people mutually know each other or m people mutually are strangers to each other. We are asked to show that $R(3, 3) \leq 6$. See **[2]** for more details and for a proof of the stronger result $R(3, 3) = 6$. (Surprisingly, $R(5, 5)$ is unknown; the best current result, according to the Wikipedia article, is $43 \leq R(5, 5) \leq 49$.)

(*No correct solutions received.*)

Week 7. *Variation on a 1989 Putnam Competition problem proposed by Matthew McMullen.*

Describe the set of points inside a square of area one that are closer to the center of the square than to any edge of the square. (**Bonus:** Find the area of this set.)

Solution: First position the square in the xy-plane so that it is centered about the origin and one of its edges is parallel to the x-axis. Let R be the region bounded by the lines $y = \pm x$ and $y = 1/2$. Then, for all points (x, y) in R, the distance to the center of our square is $\sqrt{x^2 + y^2}$ and the distance to the (nearest) edge is $1/2 - y$. Thus, in R, a point is closer to the center of the square than to any edge only if $\sqrt{x^2 + y^2} < 1/2 - y$; or, equivalently $y < 1/4 - x^2$. We may use the same method for the other three sections of the square to describe the entire region in question.

To find the area of this region, we compute

$$4 \int_{-a}^{a} (1/4 - x^2 - x)\,\mathrm{d}x = 8 \int_{0}^{a} (1/4 - x^2 - x)\,\mathrm{d}x,$$

where $a = (\sqrt{2} - 1)/2$ is the x-coordinate of the intersection point in the first quadrant of the curves $y = x$ and $y = 1/4 - x^2$. After much simplifying this area is found to be

$$\frac{4\sqrt{2}-5}{3} \approx 0.21895.$$

[*Solved by Dave Stucki (with bonus) and James Orr* (*without bonus*).]

Week 8. *Proposed by Duane Buck.*

Fix $h, w > 0$ and the point (x_2, y_2). Let R denote the rectangle centered at (x_2, y_2) with width w and height h. Let (x_1, y_1) be any point outside R, and fix $l > 0$ and $0 < \theta < \pi/2$. Find coordinates p, q, and r, where p lies on R and on the line segment connecting (x_1, y_1) and (x_2, y_2); and p, q and r make up the head of the arrow pointing from (x_1, y_1) to p with fan-out θ ending l from the tip.

Solution: We will solve the problem for $(x_2, y_2) = (0, 0)$ and describe how to find the general solution from this result. Let L_1 denote the line $y = hx/w$ and let L_2 denote the line $y = -hx/w$. Then L_1 and L_2 contain the diagonals of R and divide the plane into four regions. For (x_1, y_1) in the region below L_2 and above L_1; or, equivalently, for $hx_1/w \le y_1 \le -hx_1/w$, we have that p lies on the left-hand boundary of R and on the line containing (x_1, y_1) and the origin; namely, $y = y_1 x/x_1$. Thus,

$$p = \left(-\frac{w}{2}, -\frac{wy_1}{2x_1}\right).$$

Similarly, for (x_1, y_1) in the region below L_1 and above L_2, we have

$$p = \left(\frac{w}{2}, \frac{wy_1}{2x_1}\right);$$

for (x_1, y_1) in the region above both L_1 and L_2, we have

$$p = \left(\frac{hx_1}{2y_1}, \frac{h}{2}\right);$$

and, for (x_1, y_1) in the region below both L_1 and L_2, we have

$$p = \left(-\frac{hx_1}{2y_1}, -\frac{h}{2}\right).$$

Our next goal is to find coordinates q and r. First define $\varphi = \arctan(y_1/x_1)$. Let p_x and p_y denote the x- and y-coordinates of p, respectively. Suppose $x_1 < x_2$. Then we use right-angle trigonometry to see that

$$q = (p_x \pm l\cos(\varphi - \theta), p_y \pm l\sin(\varphi - \theta))$$

and

$$r = (p_x \pm l\cos(\varphi + \theta), p_y \pm l\sin(\varphi + \theta)),$$

where the signs are chosen so that the arrow points in the correct direction.[1] For $x_1 > x_2$, these values switch. For $x_1 = x_2$, we have $p = (0, \pm h/2)$, $r = (l\sin\theta, p_y \pm l\cos\theta)$ and $q = (-l\sin\theta, p_y \pm l\cos\theta)$.

To solve the general case, we would first translate our points $-x_2$ units horizontally and $-y_2$ units vertically [so that (x_1, y_1) would become $(x_1 - x_2, y_1 - y_2)$ and (x_2, y_2) would become $(0, 0)$]; then solve for p, q, and r as above; and then translate back to our original position.

(*No correct solutions received.*)

Week 9. *Problem from* Math Horizons *proposed by Zengxiang Tong.*

Show that

$$\int_0^1 \frac{4x^3\left(1 + x^{4(2006)}\right)}{(1+x^4)^{2008}}\,\mathrm{d}x = \frac{1}{2007}.$$

Solution: Let I denote the integral in question. Put $u = 1 + x^4$ and $w = 1 - 1/u$. Then $\mathrm{d}u = 4x^3\,\mathrm{d}x$ and $\mathrm{d}w = 1/u^2\,\mathrm{d}u$. Thus,

$$\begin{aligned} I &= \int_1^2 \frac{1 + (u-1)^{2006}}{u^{2008}}\,\mathrm{d}u \\ &= \int_1^2 u^{-2008}\,\mathrm{d}u + \int_1^2 \left(1 - \frac{1}{u}\right)^{2006} \cdot \frac{1}{u^2}\,\mathrm{d}u \\ &= \left.\frac{u^{-2007}}{-2007}\right|_1^2 + \int_0^{1/2} w^{2006}\,\mathrm{d}w \end{aligned}$$

[1] We are hand-waving here. The correct sign would likely depend on some relationship between either θ and φ or x_1 and x_2 (or both), which the reader is encouraged to find.

$$
\begin{aligned}
&= \left.\frac{u^{-2007}}{-2007}\right|_1^2 + \left.\frac{w^{2007}}{2007}\right|_0^{1/2} \\
&= \frac{1}{2007}.
\end{aligned}
$$

(*Solved by Matthew McMullen.*)

References

[1] R. Knott, http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibpuzzles2.html#jigsaw3.

[2] Wikipedia, http://en.wikipedia.org/wiki/Ramsey%27s_theorem.