The Structure of White Dwarf and Neutron Stars: Instructor Notes*

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Abstract

Some notes on the simulations are presented, along with the solutions to selected exercises. In addition, some background information on approaches to the nuclear matter equation of state are provided. Please contact the author if you have any comments or suggestions.

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Contents

I. GOALS AND TIME NEEDED

This module is a basic introduction to techniques for integrating ordinary differential equations numerically, with application to the problem of determining the equilibrium states of compact stars supported by non-thermal means, namely white dwarfs and neutron stars. Its principal goals are:

- to familiarize students with basic concepts involved in the numerical solution of ODEs;
- to provide a meaningful, physics-rich application of these techniques; and
- to allow maximum flexibility in the topics treated.

The physics involved is extremely rich, involving at some level almost every area of the discipline. Gravitational physics is essential, of course, including general relativity in the case of neutron stars, and the description of matter in the extreme states found in these objects brings in statistical mechanics, thermodynamics, quantum mechanics, and nuclear and particle physics. While this may seem daunting at first, instructors have a great deal of flexibility in choosing what is presented and how. The module should be thought of as a smorgasbord of topics, many of which can be omitted entirely if time or student background are issues. Often results can simply be motivated and used, if necessary.

The minimal physics background required will include Newtonian gravity at the level of a typical junior-level course in classical mechanics, thermodynamics, special relativity and quantum mechanics at the introductory level, and ideally some nuclear and particle physics, also at the introductory level. In my experience, students with this background will be able to follow the treatment of white dwarfs in detail, although the evaluation of the pressure in the full case (i.e., without assuming the electrons to be either highly relativistic or non-relativistic) is a bit messy. Some instructors may feel that getting through this derivation distracts the student too much from the essential physics or the numerical issues. The approximate calculations are significantly easier and should be doable with little difficulty.

The neutron star problem is naturally more involved, due to the need for relativistic corrections in the equilibrium equations and consideration of the strong nuclear interaction in the equation of state for nuclear matter. Deriving the relativistic contributions is probably beyond what can be expected from typical undergraduates, so these will most likely have to be just presented to the students. There are a number of excellent undergraduate-level texts on general relativity [1, 2], however, which can be drawn upon in presenting these results.

Regarding the nuclear equation of state, a number of options are presented, depending on student background, interests, and the time available. A simple model is given that can be used directly in the equilibrium equations. This should be no more difficult than the white dwarf problem and can be implemented quickly once that is solved. Alternatively one can take a model of the nuclear equation of state from the research literature; one such model, chosen essentially at random, is presented below. This may not seem much more interesting to the student than the simple model, however, as they are both just "black boxes."

More interesting would be to have the students themselves work through the development of an empirical model for the nuclear equation of state. Some guidance on this is given below and in the references. This will take time, however, and students will likely need to have some exposure to nuclear physics beyond the typical sophomore-level modern physics course. Familiarity with the liquid drop model, semi-empirical mass formula, etc., will be very helpful, and I would not attempt this part of the project unless students have taken an upper-level course on nuclear physics. It will also heavily weight the module towards physics, reducing the proportion of time and effort spent on computational matters to a rather low level. Depending on instructor goals, this may not be desirable.

My estimate is that two weeks of class time, assuming 3-4 hours per week in class and some work outside of it, should be sufficient to explore most of the numerical issues and solve the white dwarf problem with an approximate form for the degenerate electron equation of state. This assumes the students are already familiar with any purely computational issues such as editing, compiling, etc. If the students are well motivated one can perhaps get through the full evaluation of the equation of state in this time frame (I have done it, though I suspect it was too fast).

Three weeks would be more comfortable, especially if the neutron star problem is to be tackled. As mentioned above, once the white dwarf problem is solved it is actually fairly simple to include the relativistic corrections to the equilibrium equations and plug in a given nuclear equation of state. Pursuing the empirical model will require at least an additional week.

II. NOTES ON AND SOLUTIONS TO SELECTED EXERCISES

A. Exercises from Section III

1. *Consider the equation*

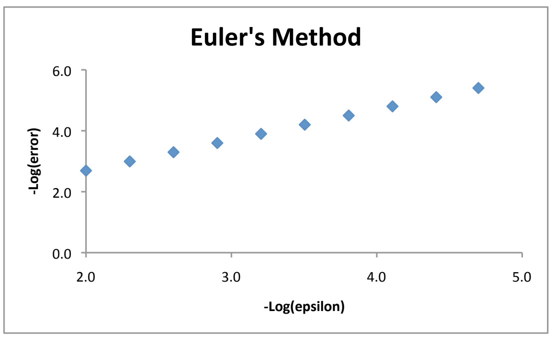
$$\frac{dy}{dx} = -xy$$

with initial condition $y(0) = 1$, which has the exact solution $y = \exp(-x^2/2)$. Study the numerical integration of this using the methods described above. In particular, verify that the errors (difference between numerical and exact solutions) decrease according to the expected power of ϵ.

A good approach here is to evaluate the solution at a fixed endpoint, say, $x = 1$, and vary the stepsize ϵ. The error should obey a power law in ϵ,

$$\text{error} \propto \epsilon^n,$$

so a plot of log(error) versus $\log(\epsilon)$ should give a line of slope n. Results for Euler's method and the second order RK algorithm are shown in fig. 1. The calculated slopes are very close to the expected values $n = 1$ and $n = 2$, respectively.



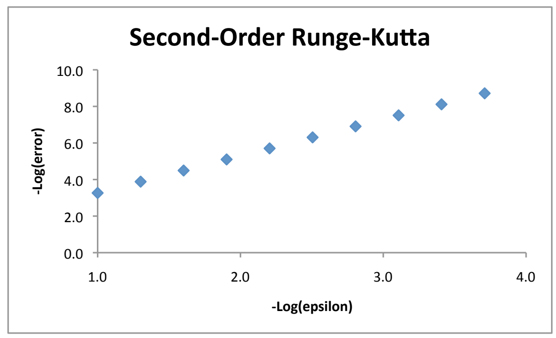


FIG. 1: Error scaling for Euler's method and the second order Runge-Kutta algorithm.

2. *Generalize one or more of the schemes presented here to solve a system of two coupled ODEs, and apply it to solve Newton's second law for the simple harmonic oscillator (with $m = 1$):*

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\omega^2 x. \end{aligned}$$

A simple criterion for accuracy is the degree to which a known integral of the motion, for example, the energy, is conserved in the evolution. Study the constancy of $2E = v^2 + \omega^2 x^2$ for the various algorithms and choices for ϵ.

As an example, here is the generalization of the second-order Runge-Kutta algorithm for a pair of ODEs involving two functions $y_1(x)$ and $y_2(x)$. We define

$$\begin{aligned} k_{11} &= \epsilon f_1(x, y_1, y_2) \\ k_{12} &= \epsilon f_2(x, y_1, y_2) \end{aligned}$$

and

$$k_{21} = \epsilon f_1(x+\epsilon/2, y_1+k_{11}/2, y_2+k_{12}/2)$$
$$k_{22} = \epsilon f_2(x+\epsilon/2, y_1+k_{11}/2, y_2+k_{12}/2),$$

that is, separate ks for each of the functions. Then

$$y_1(x+\epsilon) = y_1(x)+k_{21}$$
$$y_2(x+\epsilon) = y_2(x)+k_{22},$$

correct to $\mathcal{O}(\epsilon^3)$. The fourth-order algorithm generalizes in an analogous way.

Another measure of the error that can be used in this case is the difference between the starting value of x and the value of x at times $t_n = nT$ with $T = 2\pi/\omega$ and $n = 1, 2, 3, \ldots$

3. *Another common test of accuracy is to integrate backwards to the original starting point, using the ending x, y as the initial condition. The difference between the resulting value for y and the original initial condition then gives a measure of the overall accuracy of the result. Apply this test to the example from problem 1, for the various algorithms discussed above.*

   This criterion can be used as a measure of the error when the exact solution is not already known.

4. *Study the approximation given in eq. (3.19), by using it to solve the equation*

$$\frac{dy}{dx} = -y,$$

   with the initial condition $y(0) = 1$. This equation has the exact solution

$$y = e^{-x},$$

   of course. To start the calculation off we need both $y(0)$ and $y(\epsilon)$; you can use eq. (3.15) to obtain $y(\epsilon)$. Eq. (3.19) can then be used to generate the solution for other values of x. Does this give a reasonable approximation to the exact answer? If not, can you determine why? See the notes for further discussion of this example.

   This example is included mainly to show the sorts of things that can go wrong in practice. The stability and accuracy of integration algorithms is a large and technical subject, one to which I cannot hope to do justice here. Readers should be aware that these issues exist, and if further details are needed a reference on numerical analysis should be consulted.

B. Exercises from Section VI

1. *White dwarfs may have surface temperatures of* 10^5 *K. Why is it reasonable to treat the electron gas as* degenerate, *i.e., effectively at zero temperature (and hence in its ground state)?*

   A Fermi gas will be degenerate if the Fermi momentum p_f is much larger than the typical thermal energy $k_B T$, with k_B Boltzmann's constant. In this case thermal fluctuations can only rearrange the occupancy of states very near the Fermi surface, and the system is to all intents and purposes in its ground state. From

   $$p_f = \hbar(3\pi^2)^{1/3} n^{1/3}$$

   with $n = N/V$, we have

   $$p_f \approx \frac{\hbar(3\pi^2)^{1/3} N^{1/3}}{V^{1/3}}$$

   (compare to the estimate given in eq. (6.2)). Taking $R \sim 10^3$ km and $N \sim (1/2)M_\odot/m_N$ we then find

   $$p_f c \approx 3 \text{ MeV}.$$

   The thermal energy, on the other hand is typically

   $$k_B(10^5 \text{ K} \approx 10^{-5} \text{ MeV},$$

   which is indeed negligible compared to $p_f c$.

2. *Obtain the numerical values in MKS units for* ρ_0, M_0 *and* R_0.

   From eqs. (6.32), (6.33), (6.36) and (6.37) one finds:

   $$\begin{aligned} \rho_0 &= \frac{9.79 \times 10^8 \text{ kg m}^{-3}}{\alpha} \\ R_0 &= \alpha(4.46 \times 10^6 \text{ m}) \\ M_0 &= \alpha^2(1.09 \times 10^{30} \text{ kg}). \end{aligned}$$

   Note that with the full evaluation of the electron pressure (exercise VI.C.4), we use the same ρ_0 but a different R_0 (see eq. (8.29)). In this case we obtain

   $$R_0 = \alpha(7.73 \times 10^6 \text{ m}),$$

which then gives

$$M_0 = \alpha^2 (5.67 \times 10^{30} \text{ kg}).$$

It is perhaps worth noting that since α appears only in these relations, solutions for arbitrary α can be generated easily from the solution for any single value of α. Given a solution for $\alpha = 1$, say, one has only to multiply the density, radius and mass by α^{-1}, α, and α^2, respectively, to obtain the corresponding solution for a different α.

3. *Evaluate the electron pressure and equation of state in the non-relativistic limit, i.e., assuming $p \ll m_e$ for all electrons.*

   Here we approximate the full electron energy as

$$\sqrt{p^2c^2 + m_e^2 c^4} \approx m_e c^2 + \frac{p^2}{2m_e}.$$

   The result for the energy density is

$$\frac{E}{V} = m_e c^2 n + \frac{3\hbar^2}{10m_e}(3\pi^2)^{2/3} n^{5/3},$$

   and the pressure is given by

$$P = \frac{\hbar^2}{5m_e}(3\pi^2)^{2/3} n^{5/3}.$$

4. *Evaluate the electron pressure and equation of state in the general case, i.e., using the full relativistic expression for the electron energies. Some details of this calculation are given in the notes.*

   Solution given in student notes.

5. *Recall that our plan is to integrate numerically equations (6.38) and (6.39) starting at $\overline{r} = 0$, with initial conditions $\overline{M} = 0$ and $\overline{\rho} = \overline{\rho}_c$. Right away a complication arises, however: we cannot use $\overline{r} = 0$ to take the first step. (See the factor $\overline{r}^2$ in the denominator of eq. (6.39). Also, M will never change from zero!) Here are a couple of ways of dealing with this.*

   (a) *We can just start a short distance away from $\overline{r} = 0$, say at $\overline{r} = \epsilon$. The simplest way to do this in practice is just to stipulate that $\overline{\rho}_c$ is not quite the central density, but rather then density at $\overline{r} = \epsilon$. For consistency, then, the starting $\overline{M}$ is not zero. What should we take for $\overline{M}(\epsilon)$? You should assume that ϵ is sufficiently small that $\overline{\rho}$ is effectively constant over this range.*

This would just be the mass of a sphere of radius ϵ with density $\overline{\rho}$, except that the factor 4π has been absorbed into the rescaling. Hence

$$\overline{M}(\epsilon) \approx \frac{1}{3}\overline{\rho}\epsilon^3.$$

One can calculate this directly by integrating eq. (6.38) from 0 to ϵ, assuming $\overline{\rho} = \text{const}$.

(b) *Alternatively, we can continue to take $\overline{\rho}_c$ to be the central density and work out how all these quantities change in response to a small step in $\overline{r}$. Then we can take the first step "manually," using the differential equations to continue on once $\overline{r} \neq 0$. To implement this, imagine Taylor expanding $\overline{\rho}$ and $\overline{M}$ in powers of $\overline{r}$. For small $\overline{r}$ these will look like*

$$\overline{M} = \alpha\overline{r}^a + \cdots \tag{2.1}$$

and

$$\overline{\rho} = \overline{\rho}_c - \beta\overline{r}^b + \cdots \tag{2.2}$$

where α, a, β and b are constants, and the dots represent higher powers in $\overline{r}$ (which can be neglected for small $\overline{r}$). Determine the constants by plugging these into the equations (6.38) and (6.39) and equating powers of $\overline{r}$. Then use your results to calculate $\overline{\rho}$ and $\overline{M}$ a short distance away from the origin.

The result for $\overline{M}$ is just that given in part (a), i.e., $a = 3$ and $\alpha = \overline{\rho}_c/3$. One also finds $b = 2$ and $\beta = \overline{\rho}_c^{5/3}/6$, so that to lowest order

$$\overline{\rho} \approx \overline{\rho}_c - \frac{\overline{\rho}_c^{5/3}}{6}\epsilon^2.$$

Lest the units appear alarming, recall that these quantities are all dimensionless!

Going through this exercise for the full evaluation of the electron pressure (exercise VI.C.4) we again find $b = 2$, but now

$$\beta = \frac{\overline{\rho}_c^2}{6g(\overline{\rho}_c^{1/3})}$$

so that

$$\overline{\rho} \approx \overline{\rho}_c - \frac{\overline{\rho}_c^2\epsilon^2}{6g(\overline{\rho}_c^{1/3})}.$$

III. NOTES ON THE SIMULATION PROJECTS

A. White Dwarfs

A plot of white dwarf radius versus total mass will look like fig. 2. This plot was generated using the full equation of state and integrating with the fourth-order Runge-Kutta algorithm. The collapse of the star to $\sim$ zero size is shown in the lower right corner, from which we see that the upper limit to the mass is about $1.4M_\odot$, a good estimate for the Chandrasekhar mass.

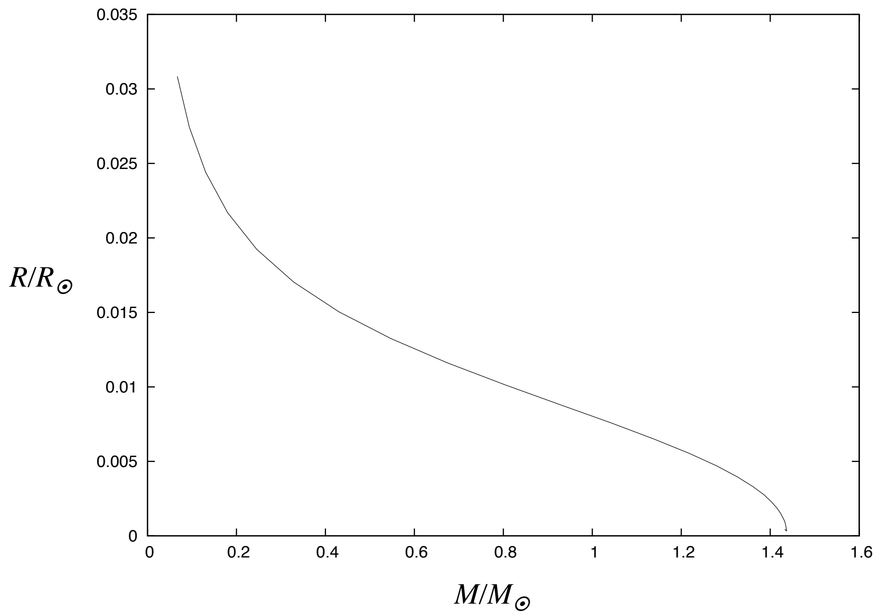


FIG. 2: Equilibrium configurations for a white dwarf with $\alpha = 0.5$ and the full equation of state. The dimensionless central density ranges from 10^{-2} (upper left) to 10^{7} (lower right).

Students should find that for larger white dwarfs the full evaluation of the degenerate electron equation of state is well approximated by treating the electrons as highly relativistic. Conversely, the non-relativistic approximation is rather poor except for small dwarfs (say M around $0.4M_\odot$ or smaller).

B. Neutron Stars

Using the equation of state for a degenerate neutron gas, the plot of mass versus radius should look like fig. 3, with the maximum mass around $0.7M_\odot$ at a radius of 9 km [7]. Note that the solutions to the left of this maximum, where the curve begins to spiral in, are actually unstable and do not exist in nature (they collapse to form black holes). The issue of stability is somewhat

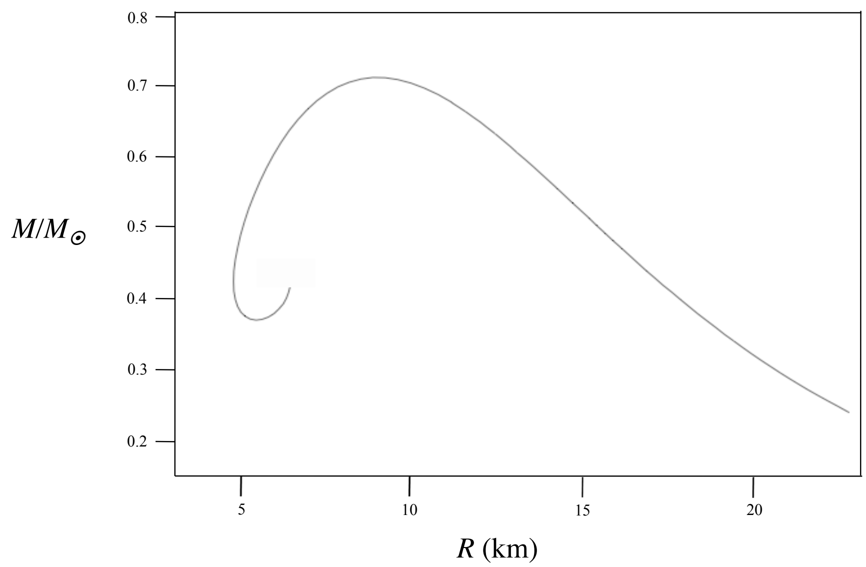


FIG. 3: Neutron star mass (solar units) versus radius (km) for the degenerate neutron gas equation of state. configrations to the left of the maximum at $R \approx 9$ km are unstable against collapse to black holes; those to the right of the maximum represent physical equilibrium states of the star.

technical and I do not attempt to discuss it here. An accessible treatment may be found in ref. [1]; many more details are available in [3].

If students add protons and electrons to the model, they should find only a small (few percent) admixture of protons, and essentially no significant change to the structure of the star.

Compared to the degenerate neutron gas equation of state, the nuclear equation of state will result in much more massive objects with comparable, or slightly larger, radii. For the maximum mass I obtain roughly $2.3M_{\odot}$, with a radius of 14 km.

IV. THE NUCLEAR MATTER EQUATION OF STATE

In this section I discuss the neutron matter equation of state in more detail. The theoretical backgrounds for the various models are beyond the scope of this module, but it is possible to motivate them somewhat by showing that they reproduce global properties of nuclei. If desired, this provides a nice tie-in to basic nuclear physics, including the semi-empirical mass formula, liquid drop model, and so on. It is also possible to have students take a purely phenomenological approach, creating their own model for the nuclear equation of state. One approach to this is discussed below.

To set the stage and establish some more notation, let us consider a system of A total nucleons

(N neutrons plus Z protons, $A = N + Z$) of mass M contained in a volume V. The nucleon number density is

$$n = \frac{A}{V}, \tag{4.1}$$

with separate proton and neutron densities

$$n_n = \frac{N}{A}, \qquad n_p = \frac{Z}{A}, \qquad n = n_n + n_p. \tag{4.2}$$

The proton-neutron asymmetry factor is defined as

$$\delta = \frac{n_n - n_p}{n_n + n_p}. \tag{4.3}$$

Symmetric nuclear matter (equal numbers of protons and neutrons) thus corresponds to $\delta = 0$, while $\delta = 1$ gives pure neutron matter.

The total energy of the nucleons will be denoted E, with the energy per nucleon E/A. The energy density is $\epsilon \equiv E/V$, with separate energy densities of neutrons and protons if desired: ϵ_n and ϵ_p with $\epsilon = \epsilon_n + \epsilon_p$. Note that, for example,

$$\frac{E}{A} = \frac{\epsilon}{n}, \tag{4.4}$$

with similar relations holding for neutron and proton densities.

Again, our first task is to compute E/A for this collection of particles, including their kinetic energies and the strong nuclear interaction energy. We then calculate the pressure from eq. (6.11) in the student notes. As a general point, the energy will be a function of n and δ, and we may expand this quantity in a Taylor series in δ about $\delta = 0$:

$$E(n, \delta) = E_0(n) + \delta^2 E_{sym}(n) + \cdots , \tag{4.5}$$

where E_0 is the energy for symmetric nuclear matter. The first-order term in the expansion vanishes on general grounds: since the strong interactions are isospin symmetric, hence invariant under exchange of protons with neutrons, the energy cannot distinguish between positive and negative δ. Furthermore, it is widely accepted that this formula, including only the quadratic correction to E, is a good approximation to E even for $\delta \approx 1$. The effects of the n-p asymmetry are thus expressed by the so-called "symmetry energy" $E_{sym}(n)$.

At this point we must appeal to a model of nuclear interactions to give us expressions for E_0 and E_{sym}. As discussed above, there are a number of plausible candidates in the literature. These basically agree well on E_0, since this is strongly constrained by lab data on real nuclei, but vary in their results for E_{sym}.

A. Skyrme model

Our first model is taken from the literature. It is a relatively simple one based on the classical Skyrme model of nuclear forces. In this approach one treats the interactions as contact (zero range) forces. It reproduces most of the global properties of ordinary nuclei, for example binding energies, charge radii, etc., very well. I present here a variant of the basic model studied in ref. [4].

In this model, the binding energy per nucleon for symmetric matter is given by

$$\frac{E_0}{A} = m_N + Cn^{2/3}(1+\beta n) + \frac{3t_0}{8}n + \frac{t_3}{16}n^{\epsilon+1}, \tag{4.6}$$

where

$$C = \frac{\hbar^2 c^2}{10M}\left(\frac{3\pi^2}{2}\right)^{2/3}, \tag{4.7}$$

$$\beta = \frac{M}{2\hbar^2 c^2}\left[\frac{1}{4}(3t_1 + 5t_2) + t_2 x_2\right], \tag{4.8}$$

and the t_i, x_i and ϵ are model parameters, specified below. The symmetry energy per nucleon is given by

$$\begin{aligned}\frac{E_{sym}(n)}{A} &= \frac{5}{9}Cn^{2/3} + \frac{10CM}{3\hbar^2 c^2}\left[\frac{t_2}{6}\left(1+\frac{5}{4}x_2\right) - \frac{1}{8}t_1 x_1\right] n^{5/3} \\ &\quad -\frac{t_3}{24}\left(\frac{1}{2}+x_3\right)n^{\epsilon+1} - \frac{t_0}{4}\left(\frac{1}{2}+x_0\right)n.\end{aligned} \tag{4.9}$$

From these expressions we can calculate the pressure as before. Let us consider first the symmetric case $\delta = 0$. Making use of

$$\frac{dE}{dV} = \frac{dE}{dn}\frac{dn}{dV}, \tag{4.10}$$

where

$$\frac{dn}{dV} = -\frac{A}{V^2} = -\frac{n}{V}, \tag{4.11}$$

we derive the useful general relation

$$P = n^2 \frac{d}{dn}\left(\frac{\epsilon}{n}\right). \tag{4.12}$$

The result for the pressure is then

$$\frac{P}{n} = \frac{2}{3}Cn^{2/3}\left(1+\frac{5}{2}\beta n\right) + \frac{3}{8}t_0 n + \frac{t_3}{16}(\epsilon+1)n^{\epsilon+1}. \tag{4.13}$$

The reader may wish to pause and verify this result.

Adding in the symmetry energy gives the pressure for the asymmetric case; we leave its evaluation as an exercise for the reader. We may assume the neutron star is composed entirely of neutrons, so that $\delta = 1$.

The parameters t_i, x_i and ϵ are determined by fitting predictions of the model to known facts about symmetric or nearly-symmetric nuclei. For example, for symmetric nuclear matter the equilibrium density is known to be $n_0 = 0.16$ fm^{-3} with a binding energy per particle at this density of $B = -16$ MeV. (Recall 1 fm $= 10^{-15}$ m). Such a fit to these and other data gives the following parameter values [4]:

t_0	t_1	t_2	t_3	x_0	x_1	x_2	x_3	ϵ
-2719.7	417.64	-66.687	15042	0.16154	-0.04799	0.027170	0.13611	0.14416

The units here are such that energies are in MeV and volume in fm^3. Note that energy density and pressure both have units MeV/fm^3. Fig. 4 below shows the binding energy per nucleon for symmetric matter in this model, while fig. 5 shows the corresponding result for pure neutrons.

Note also that the equation of state is given in parametric form, that is, the energy density $\epsilon(n) = nE(n)/A$ and pressure $P(n)$ are specified in terms of the parameter n; we do not directly calculate $P(\epsilon)$. To input this equation of state into the TOV equations, we may write

$$\frac{dP}{dr} = \frac{dP}{dn}\frac{dn}{dr}, \tag{4.14}$$

and divide by the factor dP/dn as before. The result is an equation for the radial variation of n. Since P is given as a function of n its derivative may be computed readily, and in addition ϵ may everywhere be written in terms of n. Thus the system of equations gives $n(r)$ and the total mass-energy contained within r. Integrating outward from the center gives the density profile of the star, and when $n = 0$ we have reached the edge.

B. Empirical model for the equation of state

An alternative to simply presenting the above equation of state is to develop a simple empirical model tied to basic nuclear properties. In this section I sketch one approach to this problem, inspired by material in refs. [5, 6].

The goal as always is to obtain an expression for E/A, the energy per nucleon. We shall continue to make use of eq. (4.5), and consider separately the energy of symmetric nuclear matter E_0 and the symmetry energy E_{sym}.

Consider first the energy of symmetric nuclear matter. There are a number of basic constraints that our model should respect. For example, the equilibrium (number) density of nuclear matter is $n_0 = 0.16\ \mathrm{fm}^{-3}$ and the binding energy per nucleon at this density is about $B = 16$ MeV. Thus our model E_0, regarded as a function of n, should have a minimum at $n = n_0$, with the value $E_0(n_0) - m_N c^2 = -16$ MeV. In addition, the nuclear "compressibility" K, defined by

$$K(n) = 9\frac{dP(n)}{dn}, \tag{4.15}$$

is related to the degree of curvature at the minimum (it involves the *second* derivative of ϵ with respect to n). We shall denote the value at the minimum by $K(n_0) \equiv K_0$. This quantity is not so well known experimentally, but K_0 is probably somewhere in the range 200 to 400 MeV.

We can also identify some specific contributions to E_0. The simplest is just the rest energy of each nucleon, $m_N c^2$. In addition the nucleons have kinetic energy, which we can model using the machinery from the degenerate Fermi gas. Let us assume that the nucleons are essentially non-relativistic, so that their kinetic energies are given by $p^2/2m$. In the symmetric case, the protons and neutrons have separate Fermi seas with identical Fermi momenta:

$$p_f^3 = 3\pi^2\hbar^3(n/2), \tag{4.16}$$

where $n_p = n_n = n/2$. The total kinetic energy of either protons or neutrons is then

$$KE = 2V\int_0^{p_f} \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{2m_N}. \tag{4.17}$$

Evaluating the integral gives

$$\frac{KE}{A} = \frac{3}{5}\frac{p_f^2}{2m_N}. \tag{4.18}$$

We may interpret this as the average kinetic energy per nucleon in the system. Recall that $p_f \propto n^{1/3}$, so this contribution to E_0 behaves as $n^{2/3}$.

What else shall we add to the model E_0? Basically, enough to produce a well with a minimum at n_0 and the correct binding energy and compressibility. Since we have three input data values, our model form should contain three free parameters. We choose essentially the simplest form that will give what we require, namely

$$\frac{E_0(n)}{A} = m_N c^2 + \frac{3}{5}\frac{p_f^2}{2m_N} + C(n/n_0) + D(n/n_0)^\alpha. \tag{4.19}$$

The three parameters to be fit to the data are C, D and α. In detail the constraints are:

1. that $E_0(n)$ have a minimum at $n = n_0 = 0.16\ \mathrm{fm}^{-3}$;

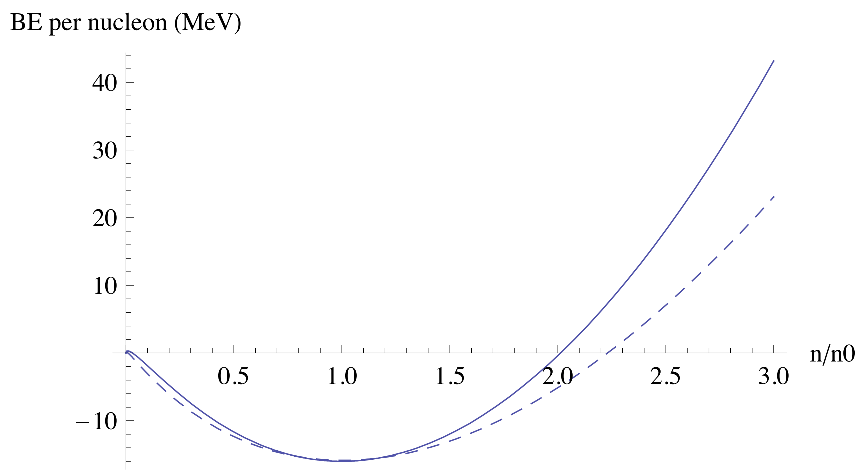


FIG. 4: Binding energy per nucleon ($E_0/A - m_N$) versus n/n_0 for symmetric nuclear matter given by the empirical model (solid) and Skyrme model (dashed). The empirical model uses $K_0 = 300$ MeV.

2. that $E_0(n_0) - m_N c^2 = -16$ MeV; and

3. That K_0 have some specified value between 200 and 400 MeV.

Imposing these constraints on eq. (4.19) I find, assuming $K_0 = 300$ MeV,

$$\alpha = 1.64, \qquad C = -74.8, \qquad D = 36.7. \tag{4.20}$$

(Readers should experiment with other choices for K_0.) Note that the kinetic energy term may be written in the form

$$\frac{3}{5}\frac{p_f^2}{2m_N} = KE_0(n/n_0)^{2/3}, \tag{4.21}$$

where KE_0 is the average kinetic energy per nucleon at the equilibrium density:

$$KE_0 = \frac{3}{5}\frac{(3\pi^2 n_0/2)^{2/3}}{2m_N}. \tag{4.22}$$

Its numerical value is $KE_0 = 22.1$ MeV. A plot of the resulting $E_0(n)/A - m_N$ is shown in fig. 4, along with the corresponding result from the Skyrme model. The agreement can be improved by choosing a smaller value of K_0 in the empirical model.

Next we turn to E_{sym}. This object is not well known and we have a great deal of freedom in designing it. The main experimental fact that is relevant is that the bulk symmetry energy at the equilibrium density, $E_{sym}(n_0)$, is around 30 MeV. (It is thought to be in the range 25-35 MeV. We have chosen the central value but readers may wish to study the effects of varying this input.)

Now, part of E_{sym} can be modeled directly. Protons and neutrons are non-identical particles so they have separate Fermi seas. In the symmetric case each sea is filled to the same level and so $p_{f,n} = p_{f,p}$. In the asymmetric case, however, this will no longer be so. If we consider the extreme case of pure neutron matter, with the same total number of nucleons as in the symmetric case, then there are no protons and the neutron sea will be filled to a higher Fermi momentum; hence the average kinetic energy per particle is higher than for symmetric matter at the same density.

Let us work out the average kinetic energy per particle as before. Again we assume the nucleons are non-relativisitic, so that the average KE for neutrons and protons, respectively, are

$$\frac{KE_n}{N_n} = \frac{3}{5}\frac{p_{f,n}^2}{2m_N} \tag{4.23}$$

and

$$\frac{KE_p}{N_p} = \frac{3}{5}\frac{p_{f,p}^2}{2m_N}, \tag{4.24}$$

where $p_{f,n}^3 = 3\pi^2\hbar^3 n_n$, etc. The average kinetic energy per particle is then

$$\frac{KE_n + KE_p}{A} = \frac{3}{5}\frac{p_{f,n}^2}{2m_N}\left(\frac{N_n}{A}\right) + \frac{3}{5}\frac{p_{f,p}^2}{2m_N}\left(\frac{N_n}{A}\right), \tag{4.25}$$

where $N_n/A = n_n$, etc. Using

$$n_n = n\left(\frac{1+\delta}{2}\right), \qquad n_p = n\left(\frac{1-\delta}{2}\right), \tag{4.26}$$

this can be written in the form

$$\frac{KE}{A} = \frac{3}{5}\frac{(3\pi^2\hbar^3 n/2)^{2/3}}{2m_N} \cdot \frac{1}{2}\left[(1+\delta)^{5/3} + (1-\delta)^{5/3}\right]. \tag{4.27}$$

As a check, the reader can verify that this reduces to our previous result in the symmetric case $\delta = 0$.

Now the difference between this and the symmetric kinetic energy should be included in E_{sym}. Inspired by eq. (4.5), let us Taylor expand the above result in δ about $\delta = 0$. The result is

$$\frac{KE}{A} = \frac{3}{5}\frac{(3\pi^2\hbar^3 n/2)^{2/3}}{2m_N}\left[1 + \frac{5}{9}\delta^2 + \cdots\right]. \tag{4.28}$$

The term "1" in square brackets is just the symmetric energy, so the part to be included in E_{sym} is

$$\frac{\Delta KE}{A} = \frac{3}{5}\frac{(3\pi^2\hbar n/2)^{2/3}}{2m_N} \cdot \frac{5}{9}\delta^2. \tag{4.29}$$

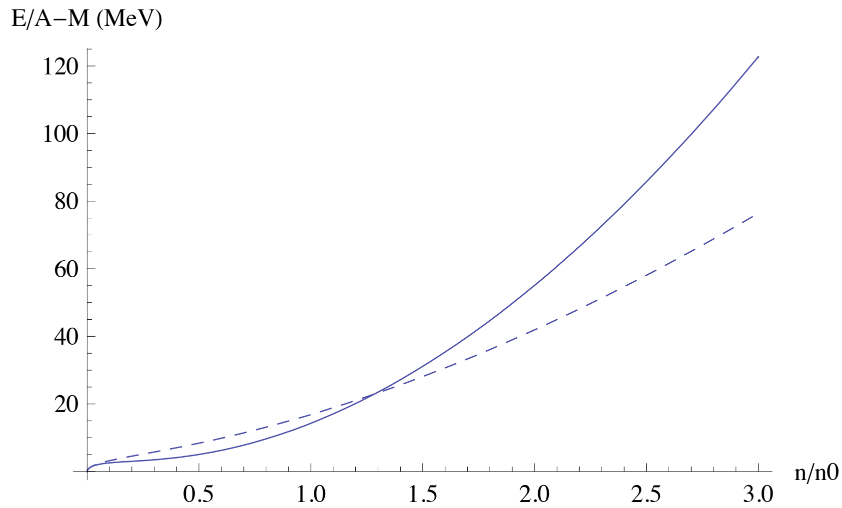


FIG. 5: Energy per nucleon ($E/A - m_N$) versus n/n_0 for pure neutron matter given by the empirical model (solid) and Skyrme model (dashed). The empirical model uses $K_0 = 300$ MeV.

For the pure neutron case ($\delta = 1$) it is reasonable to just retain the entire KE without Taylor expanding; the result is to replace the 5/9 by $(2^{2/3} - 1)$ in eq. (4.29). This makes about a 5% difference for pure neutron matter. Retaining the δ dependence allows stars with some admixture of protons to be considered, however.

We can re-write ΔKE as

$$\frac{\Delta KE}{A} = \frac{5}{9}\delta^2 KE_0 (n/n_0)^{2/3}, \tag{4.30}$$

where $KE_0 = 22.1$ MeV as before. The model symmetry energy is then

$$E_{sym}(n) = \frac{5}{9} KE_0 (n/n_0)^{2/3} + S(n), \tag{4.31}$$

where we have lumped everything else into a remainder function $S(n)$. This function is poorly known and we shall simply assume $S(n) = An$ with A chosen to satisfy $E_{sym}(n_0) = 30$ MeV. This gives $A = 111$ MeV·fm^3. Readers are encouraged to try alternate forms.

The resulting energy per nucleon for pure neutron matter is shown in fig. 5, along with the prediction of the Skyrme model, for densities up to $10n_0$; this is about the largest density encountered at the core of a neutron star.

From this result the pressure may be calculated as before. The result is another parametric equation of state for use in the TOV equation.

C. Some Additional Exercises

1. What is p_f at the equilibrium density n_0? Does it make sense to treat the nucleons non-relativistically?

2. Derive the pressure $P(n)$ for pure neutron matter in the Skyrme model.

3. Derive the pressure for asymmetric nuclear matter in the empirical model for the nuclear energy density. Compare the Skyrme and empirical model equations of state. Which do you expect to result in larger neutron stars?

4. In the empirical model, make a parametric plot of P versus ϵ. Fit the result to a polytrope of the form

$$P = K\epsilon^{\gamma},$$

determining the best values for γ and K. This can also be used as an especially simple non-parametric EOS in the simulations below.

V. SAMPLE PROGRAMS

A set of sample programs, written in C, is distributed with the module. Typically they take some command line arguments, with some other parameters hard-wired via `#define` statements. Comments in the programs explain the arguments.

The programs included are:

1. `euler1.c`

   Solves the equation from exercise III.D.1 using Euler's method.

2. `rk1.c`

   Solves the equation from exercise III.D.1 using the second order Runge-Kutta algorithm.

3. `rk2.c`

   Solves the equation from exercise III.D.2 (harmonic oscillator) using the second order Runge-Kutta algorithm generalized to two coupled equations.

4. `bad4.c`

   Solves the equation from exercise III.D.4 using the "bad" algorithm from the text.

5. `wd.c`

   Equilibrium configurations of a white dwarf. Uses the full degenerate electron gas equation of state with fourth-order Runge-Kutta integration.

6. `wd-relpoly.c`

   Equilibrium configurations of a white dwarf. Uses the relativistic polytropic equation of state with fourth-order Runge-Kutta integration.

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