THE TRAVELING SALESMAN PROBLEM

Example/Motivation  Julienne Ward has just been promoted as Sales Director for her company. She lives in Orlando, Florida, and oversees regional offices in Orlando, Atlanta, Memphis and New Orleans. She frequently needs to fly to each regional office for meetings. She would like to determine the least expensive route to visit each city one time and then return to Orlando. To help analyze this problem Julienne used the internet to find the least expensive one–way fares offered between each of the four cities.

<table>
<thead>
<tr>
<th></th>
<th>Orlando</th>
<th>Atlanta</th>
<th>Memphis</th>
<th>New Orleans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orlando</td>
<td>*</td>
<td>$67</td>
<td>$95</td>
<td>$69</td>
</tr>
<tr>
<td>Atlanta</td>
<td>$67</td>
<td>*</td>
<td>$57</td>
<td>$68</td>
</tr>
<tr>
<td>Memphis</td>
<td>$95</td>
<td>$57</td>
<td>*</td>
<td>$99</td>
</tr>
<tr>
<td>New Orleans</td>
<td>$69</td>
<td>$68</td>
<td>$99</td>
<td>*</td>
</tr>
</tbody>
</table>

Problems like the one above are called *traveling salesman problems*. To help analyze traveling salesman problems, the cost or distances are indicated along each edge of the graph. Such a graph is called a *weighted graph*. 
History of the Traveling Salesman Problem (TSP)

In the 1857 the Irish mathematician Sir William Rowan Hamilton was interested in find a solution to the Icosian Game. The game’s object is finding a cycle along the edges of a dodecahedron such that every vertex is visited a single time, and the ending point is the same as the starting point.

The general form of the TSP appears to have been first studied by mathematicians during the 1930s in Vienna and at Harvard, notably by Karl Menger, who defines the problem. The problem is easy to state but very difficult to solve.

Applications of the Traveling Salesman Problem (TSP)

1. **Routing around Cities** (plane routing, telephone routing)

2. **Computer Wiring** (connecting together computer components using minimum wire length)

3. **Genome Sequencing** (arranging DNA fragments in sequence)
**Definition**  A Hamilton circuit of a graph is an ordered sequence of edges of the edges of the graph such that

1. each vertex appears exactly twice in the sequence,
2. the second vertex of each edge in the sequence is the first vertex of the next edge,
3. the first and last vertices are the same

![Diagram of a graph](image)

**Example**  Which of the following graphs have a Hamiltonian circuit?

![Graphs G1, G2, G3](image)

In an **Euler circuit** the path followed must include **every edge** and must begin and end at the same vertex.

A connected graph has an Euler circuit if every vertex has even degree.

In a **Hamilton circuit**, the path followed must include **every vertex** and must begin and end at the same vertex, but unlike the Euler circuit, a Hamilton circuit does not have to include every edge.

There is no result to tell us under what conditions we have a Hamilton circuit.
Our goal in a traveling salesman problem is to find the least expensive or shortest way to visit each city once and return home. In terms of graph theory, our goal is to find the Hamilton circuit with the lowest associated cost or distance. We will discuss two methods for determining such a Hamilton circuit.

The Brute Force Method for Solving the Traveling Salesman Problem

1. List all possible Hamilton circuits for a graph
2. Determine the cost (distance) associated with each of these Hamilton circuits.
3. Choose the Hamilton circuit with the lowest cost (distance).

Example  Recall the example on the first page.
Example  Suppose that you want to visit the state capitals of all 50 states. How many different itineraries are possible?

In 1954 Dantzig, Fulkerson, and Johnson, solved the problem for 49 state capitals (Alaska and Hawaii became states in 1959).

Proctor and Gamble ran a contest in 1962. The contest required solving a TSP on a specified 33 cities. There was a tie between many people who found the optimum solution.

Groetschel (1977) found the optimal tour of 120 cities from what was then West Germany.

Padberg and Rinaldi (1987) found the optimal tour of 532 AT&T switch locations in the USA.

Padberg and Rinaldi (1987) found the optimal tour through a layout of 2,392 points obtained from Tektronics Incorporated.

Applegate, Bixby, Chvatal, and Cook (1994) found the optimal tour for a 7,397–city TSP that arose in a programmable logic array application at AT&T Bell Laboratories.
Nearest Neighbor Method of Determining an Approximate Solution to a Traveling Salesman Problem

1. Identify the starting vertex

2. Of all the edges attached to the starting vertex, choose the edge that has the smallest weight. Travel along this edge to the second vertex.

3. At the second vertex, choose the edge that has the smallest weight that does not lead to a vertex already visited. Travel along this edge to the third vertex.

4. Continue this process, each time moving along the edge with the smallest weight until all vertices are visited.

5. Travel back to the original vertex.

Example  Use this method to find a solution to the first example.
Example A traveling salesman wants to start from Detroit, visit each of the following cities exactly once and return to its starting point. In which order should he visit these cities to travel the minimum total distance?
Find the route with the least total airfare that starts from Boston, visits each of the cities on the following graph, and ends at Boston.

Find the route with the least total airfare that starts from New York visits each of the cities on the above graph, and ends at New York.
Homework

1. What is a Hamilton circuit?

2. What is the difference between a Hamilton circuit and an Euler circuit?

3. Dale Klitzke is a milk truck driver. Dale has to start at the processing plant and pick up milk on seven different farms. In how many ways can Dale visit each farm only once and return to the processing plant?

4. Christina is searching for a new job. She lives in Shreveport, Louisiana, and has interviews in Barrow, Alaska; Tucson, Arizona; and Rochester, New York. The cost of the one-way flights between these four cities are as follows: Shreveport to Barrow costs $855, Shreveport to Tucson costs $803, Shreveport to Rochester costs $113, Barrow to Tucson costs $393, Barrow to Rochester costs $337, and Tucson to Rochester costs $841. Use either method to find the least expensive route for Christina to travel to each city and return home to Shreveport. What is the minimum cost she can pay?

5. Jim Bowman leaves in Boston and wishes to visit cranberry farms in the following locations: Madison, WI; Princeton NJ; Salem, OR; and Walla Walla, WA. The one-way flight prices are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Boston</th>
<th>Madison</th>
<th>Princeton</th>
<th>Salem</th>
<th>Walla Walla</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boston</td>
<td>$131</td>
<td>$256</td>
<td>$298</td>
<td>$576</td>
<td></td>
</tr>
<tr>
<td>Madison</td>
<td>$131</td>
<td>$154</td>
<td>$356</td>
<td>$970</td>
<td></td>
</tr>
<tr>
<td>Princeton</td>
<td>$256</td>
<td>$154</td>
<td>$353</td>
<td>$1164</td>
<td></td>
</tr>
<tr>
<td>Salem</td>
<td>$298</td>
<td>$356</td>
<td>$353</td>
<td></td>
<td>$179</td>
</tr>
<tr>
<td>Walla Walla</td>
<td>$576</td>
<td>$970</td>
<td>$1164</td>
<td>$179</td>
<td>*</td>
</tr>
</tbody>
</table>

(a) Use the Nearest Neighbor Method to approximate the least expensive route for Jim to travel to each city and return to Boston. Give the cost of the route determined.

(b) Randomly select another route for Jim to travel from Boston to the other cities and return to Boston and then compute the cost of this route. Compare this cost with the cost found in part (a).

6. The distances in miles between four cities are shown in the table

<table>
<thead>
<tr>
<th></th>
<th>Pittsburgh</th>
<th>Philadelphia</th>
<th>Baltimore</th>
<th>Washington</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pittsburgh</td>
<td>–</td>
<td>305</td>
<td>244</td>
<td>245</td>
</tr>
<tr>
<td>Philadelphia</td>
<td>305</td>
<td>–</td>
<td>100</td>
<td>136</td>
</tr>
<tr>
<td>Baltimore</td>
<td>244</td>
<td>100</td>
<td>–</td>
<td>38</td>
</tr>
<tr>
<td>Washington</td>
<td>245</td>
<td>136</td>
<td>38</td>
<td>–</td>
</tr>
</tbody>
</table>
(a) Draw a complete weighted graph for the information in the table

(b) Find the shortest route from Pittsburgh to all other cities and back to Pittsburgh using the Brute Force Method

(c) Use the Nearest Neighbor Method to find an approximation of the shortest circuit starting and ending at Pittsburgh. How do the route and mileage compare to the actual optimal solution?

7. Air fares (in dollars) between cities are shown in the table below:

<table>
<thead>
<tr>
<th></th>
<th>New York</th>
<th>Cleveland</th>
<th>Chicago</th>
<th>Baltimore</th>
</tr>
</thead>
<tbody>
<tr>
<td>New York</td>
<td>–</td>
<td>375</td>
<td>450</td>
<td>200</td>
</tr>
<tr>
<td>Cleveland</td>
<td>375</td>
<td>–</td>
<td>250</td>
<td>300</td>
</tr>
<tr>
<td>Chicago</td>
<td>450</td>
<td>250</td>
<td>–</td>
<td>325</td>
</tr>
<tr>
<td>Baltimore</td>
<td>200</td>
<td>300</td>
<td>325</td>
<td>–</td>
</tr>
</tbody>
</table>

(a) Draw a complete weighted graph for the information in the table

(b) Find the cheapest route from Chicago to all other cities and back to Chicago using the Brute Force Method

(c) Use the Nearest Neighbor Method to find an approximation of the cheapest circuit starting and ending at Chicago. How do the route and cost compare to the actual optimal solution?